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O

AN INTRODUCTION

TO

NATURAL PHILOSOPHY

DESIGNED AS A TEXT-BOOK IN

PHYSICS

FOR THE USE OF STUDENTS IN COLLEGE

BY

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FOURTH REVISED EDITION

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OLMSTED'S COLLEGE PHILOSOPHY

FOURTH REVISED EDITION

PREFACE.

IT is now nine years since Olmsted's Natural Philosophy was last revised. In this time many changes have been made in the technical nomenclature of Physics, many improvements in the methods of presentation of complicated portions of the science have been published by experienced educators, and, above all, the whole subject of Electricity and Magnetism has outgrown its former apparel. For the present revision the whole book has been carefully gone over. A comparison between the respective Tables of Contents of the old and new editions will indicate the thoroughness of the work. The chief efforts of the revising editor have been spent, however, in rewriting the parts treating of Electricity and Magnetism.

The fact that the electrical units are based upon the C. G. S. units has necessitated the introduction of the latter in the Mechanics. For the sake of simplicity, these units have been also introduced into the treatment of Specific Gravities. The subjects Force, Energy, and Work in Part I., Wave Motions in Part II., Organ Pipes in Part IV., Spectrum Analysis and Interference of Light Waves in Part V., have been almost entirely rewritten. A simple discussion of Lens Images has been inserted in place of the more complex one of the last revision. An explanation of the cause of the colors, yielded by double-refracting crystals, has been introduced in the treatment of Polarized Light. Additions have also been made to the chapter on Elasticity.

Under the subject of Heat, the chief alterations have resulted from the introduction of the Gram-Calorie as the unit of heat.

In writing the Electricity and Magnetism it has been the aim of the author to present clearly the principles of the subject as they are viewed at the present time. To do this in the limits of 102 pages has necessitated the omission of descriptions of many experiments which are valuable only as they happen to correspond with instruments occupying space in many phys-

ical cabinets. Extended description of apparatus has been avoided. A few striking experiments have been described, but the choice of demonstration has been left largely to the instructor or professor in charge. Many new drawings have been made. These are chiefly in outline, as being thus best able to perform their functions in a text-book.

POLYTECHNIC INSTITUTE OF BROOKLYN,
June, 1891.

PUBLISHER'S NOTE.

SHELDON'S ELECTRICITY. Chapters on Electricity, an introductory text-book for students in college, by Samuel Sheldon, Ph.D. (Würzburg), Professor of Physics and Electrical Engineering, Polytechnic Institute of Brooklyn.

These chapters, prepared for and included in the Fourth Revised Edition of Olmsted's College Philosophy, are also published in a separate volume of 102 pages, octavo, cloth. Price, $1.25.

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NATURAL PHILOSOPHY.

INTRODUCTION.

Art. 1. Definition.—*Physics, or Natural Philosophy, is the science of matter and energy.* The broadness of this definition makes the science include three special departments, which, from the great number of closely related facts belonging to them, are considered as distinct sciences. They are Biology, which treats of matter endowed with the principle of life; Chemistry, treating of the inner mechanism of ultimately divided matter; and Astronomy, treating of gross matter in the form of worlds.

Extensive developments have rendered it necessary to limit the domain of Physics, until it is now popularly considered as treating of those things which pertain to Mechanics, Sound, Heat, Light, Electricity, and Magnetism.

2. Definitions Relating to Matter.—

A *Body* is a separate portion of matter, whether large or small.

An *Atom* is a portion of matter so small as to be indivisible.

A *Particle* denotes the smallest portion which can result from division by mechanical means, and consists of many atoms united together.

A *Molecule* is the smallest portion of any substance which can exist in a free state, and is made up of atoms.

Mass is the quantity of matter in a body, and is usually measured by its weight.

Volume signifies the space occupied by a body.

Density expresses the relative mass contained within a given volume. Thus, if one body has twice as great a mass within a certain volume as another has, it is said to have twice the density.

Pores are the minute portions of space within the volume of a body which are not filled by the material of that body. All matter is porous, some kinds in a greater and some in a less degree.

3. Properties of Matter.—

(1.) *Extension.*—Every portion of matter, however small, has length, breadth, and thickness, and thus occupies space. This is its extension.

(2.) *Impenetrability.*—While matter occupies space it excludes all other matter from it, so that no two atoms can be in exactly the same place at the same time. This property is called impenetrability.

The two foregoing are often called *essential* properties, because we cannot conceive matter to exist without them.

(3.) *Divisibility.*—Matter is *divisible* beyond any known limits. After being divided, as far as possible, into particles by mechanical methods, it may be still further reduced by chemical action to atoms, which are too small to be in any way recognized by the senses.

(4.) *Compressibility.*—Since pores exist in all matter, it may be compressed into a smaller volume. Hence all matter is *compressible*, though in very different degrees.

(5.) *Elasticity.*—After a body has suffered compression, it shows, in some degree at least, a tendency to restore itself to its former volume. This property is called elasticity. A body is said to be *perfectly elastic* when the force by which it recovers its size is equal to that by which it was before compressed. The word elasticity is used generally in a wider sense than is given in the above definition, namely, the tendency which a body has to recover its original *form*, whatever change of form it may have previously received. Thus, if a body is stretched, bent, twisted, or distorted in any other way, it is called *elastic*, if it tends to resume its form as soon as the force which altered it has ceased. *Torsion* is the name of the elastic force which tends to untwist a thread or wire when it has been twisted.

(6.) *Attraction.*—This is the general name used to express the universal tendency of one portion of matter toward another. It receives different names, according to the circumstances in which it acts. The attraction which binds together atoms of different kinds, so as to form a new substance, is called *affinity*, and is discussed in Chemistry; that which unites particles, whether simple or compound, so as to form a body, is called *cohesion*; the clinging of two kinds of matter to each other, without forming a new substance, is called *adhesion*; and the tendency manifested by masses of matter toward each other, when at sensible distances, is called *gravitation*.

(7.) *Inertia.*—This is also a universal property of matter, and signifies its tendency to continue in its present condition as to

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motion or rest. If at rest, it cannot move itself; if in motion, it cannot stop itself or change its motion, either in respect to direction or velocity.

4. The Absolute Units.—Physics is essentially a science of measurement. Everything treated in it is the direct result of observation. All its laws have been derived from patient measurement.

If all lengths, from the diameter of an atom to the distance between us and the farthest star, could be directly measured; if all masses, from the weight of an atom to the weight of the sun, could be accurately determined, and if every conceivable duration of time could be measured, then all physical measurements would demand but three units, viz., a unit of length, a unit of mass, and a unit of time. This peculiar property of these magnitudes earns for their respective units the name *Absolute Units*.

The absolute unit of *length* is the *centimeter*; of *mass*, the *gram*; of *time*, the *second*.

Other units, whose determination involves the use of more than one absolute unit, are called *derived units*. Thus, velocity requires that length and time be determined.

Temperature, which we now measure by thermometers, would, in all probability, be measured as a length, in absolute units. This length would be the distance through which a molecule vibrates.

PART I.

MECHANICS.

CHAPTER I.

MOTION AND FORCE.

5. Classification of Motions.—Motion is change of place, and may be classified as follows:

Uniform Motion, when equal spaces are passed over in equal times;

Accelerated Motion, when the spaces described in equal times become continually greater;

Retarded Motion, when the spaces described in equal times become continually less.

In the last two cases, if the increments or decrements of space for equal times are equal, the motion is *uniformly* accelerated or retarded.

The space described by a body moving uniformly in the unit of time is termed the *Velocity*. Accordingly the unit velocity would be when a unit's length was passed in a unit time, *e.g.*, centimeter per second or foot per second.

6. Uniform Motion.—When motion is uniform, the number of centimeters described in one second, multiplied by the number of seconds, obviously gives the whole space. Let s = space, t = time, and v = velocity; then $s = t\,v$; whence

$$v = \frac{s}{t}.$$

7. Questions on Uniform Motion.—

1. A ball was rolled on the ice with a velocity of 780 centimeters per second, and moved uniformly 21 seconds; what *space* did it describe? *Ans.* 16,380 cms.

2. A steamboat moved uniformly across a lake 17 miles wide at

the rate of 20 feet per second; what *time* was occupied in crossing? *Ans.* 1h. 14m. 48s.

3. On the supposition that the earth describes an orbit of 600,000,000 of miles in 365¼ days, with what *velocity* does it move per second? *Ans.* 19 miles, nearly.

4. Three planets describe orbits which are to each other as 15, 19, and 12, in times which are as 7, 3, and 5; what are their *relative velocities?* *Ans.* 225, 665, and 252.

8. Momentum.—The product of the mass of a body and its velocity is called *Momentum*. Thus let $k =$ momentum, $m =$ the mass, and $v =$ the velocity, and we have $k = m\,v$, $m = \frac{k}{v}$, and $v = \frac{k}{m}$.

If the momentum of one body equals that of another, then, since $k = k'$, $m\,v = m'\,v'$, $\therefore\ m : m' :: v' : v$. That is, in order that the momenta of two bodies should be equal, their masses must vary inversely as their velocities.

9. Questions on Momentum.—

1. A cannon-ball weighing 12 kilos., with a velocity of 820 meters per second, hits a ship with what momentum?

2. A ball weighing 10 grams is fired into a log weighing 9,990 grams, suspended so as to move freely, and imparts a velocity of 1 meter per second. Assuming that the log and ball have a momentum equal to the previous momentum of the ball alone, required the velocity of the ball? *Ans.* 1 kilometer per second.

3. Suppose a comet, whose velocity is 1,000,000 miles per hour, has the same momentum as the earth, whose velocity is 19 miles per second; what is the ratio of their masses? *Ans.* 1 : 14.6.

4. Two railway cars have their quantities of matter as 7 to 3, and their momenta as 8 to 5; what are their relative velocities? *Ans.* As 24 to 35, or nearly 5 to 7.

10. Force.—*Force is that which tends to produce, alter, or destroy motion.*

Gravity, friction, explosions, elasticity, and magnetic attractions are forces.

We say the *tendency* of a force is to produce motion. It does not always do so, for the body acted upon may be rigidly restrained from moving by an equal opposite force, which comes into existence only as the first force commences to act. Thus one may exert a force to pull a nail out of a hard-wood plank. The instant the

force of the pull is exerted an opposite equal force of friction is generated.

According to the duration of time which a force acts it is classed as

An Impulsive force, when acting for an inappreciable length of time, or

A Continued force, when acting for a sensible length of time.

The blows of a hammer, explosions, and electric discharges are impulsive forces, while gravitation, magnetic attractions, and winds are continued forces.

When the strength of a force remains unaltered with the time it is called a *Constant* force. Such a one is gravity.

11. Motions Produced by Force.—If a free body at rest be acted upon by a single force we find that

An *impulsive* force causes *uniform* motion, and

A *continued* force causes *accelerated* motion. If the continued force be constant, the resulting motion is uniformly accelerated.

Of course it is impossible to practically realize the motion resulting from a single force. A ball sent by an impulsive discharge is subject not only to the influence of gravity, but also to the force of friction from the air. Even gravity has to work against the force of friction.

12. Measure of Force.—In order to measure such an intangible thing as force, we must look to the effects which it has, and measure them.

Impulsive Forces.—In the case of an impulsive force this is easy, for it gives to the object acted upon a uniform velocity. Both the body and the velocity must be considered. Experiments with the *same force*, but different bodies, show that the greater the mass of the body the less the velocity which will be imparted to it. Furthermore, it will be found that, with the same force, the product of mass and velocity will be constant. Whence, representing force by F, we have

$$F \propto m\,v,$$

But (Art. 8) $$k \propto m\,v.$$

Hence *an impulsive force is measured by the momentum it produces or destroys.*

Constant Forces.—The effects of a constant force are more numerous. A constant force like the pull of a magnet (through short distances) can be made to extend an elastic spring by a measurable amount and maintain the extension. It can be made

to oppose the constant force of gravity, as by attracting one end of a balance. It can be made to impart a uniformly accelerated motion to a given mass.* All these methods are practically employed in the measurement of constant forces. The elasticity of a spring is, however, too complicated to be used in obtaining a unit of force. Gravity differs (Art. 16) at various parts of the earth. Hence use is made of the accelerated motion imparted to a given mass.

Of two different constant forces, working for the same length of time, the stronger will give a greater acceleration to the same mass, and will give the same acceleration to a greater mass. Letting a = acceleration, this is expressed by

$$F \propto m\,a.$$

To get our unit of force we change to an equality,

$$F = m\,a.$$

Making m and a units we have for the unit force one that will produce a unit acceleration on a unit mass. Unless we know what a unit acceleration is, we are still at sea.

Acceleration is gain of velocity, *i.e.*, difference between the velocity at the beginning of a period of time and the velocity at the end of that period. The longer the period the greater the gain, hence a unit period (second) should be taken. Further, it is well to consider the first velocity as zero, *i.e.*, the body to be at rest when first acted upon. Thus a unit acceleration is a unit velocity gained in a unit time. As an equality,

$$a = \frac{v}{t}.$$

Substituting in the equation above we have

$$F = \frac{m\,v}{t}.$$

Expressing all these quantities in absolute units, we have the absolute unit of force.

The Dyne is that force which, acting upon one gram for one second, will produce a velocity of one centimeter per second.

To measure an impulsive force in this manner would necessitate the measurement of the infinitely short time that it acts. It really produces an accelerated motion while it acts, and the uniform velocity which we observe is the gain during that short time.

(The force exerted by the earth upon one pound of matter is oftentimes taken as the unit of force.)

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* On the measurement of constant forces by the method of oscillations, see Art. 163.

QUESTIONS ON FORCE.—

1. How many dynes are required to set a mass weighing 50 kilos. in motion with a velocity of 12 meters per second, the force being supposed to act for precisely one second? *Ans.* 60,000,000.

2. How many dynes are required to move a gram 9.81 meters per second, the force acting for 1 second? Acting for 2 seconds? *Ans.* 981; 490.5.

3. Ten dynes act, for a second, on a kilogram. What is its velocity?

13. The Three Laws of Motion.—All the phenomena of motion in Mechanics and Astronomy are found to be in accordance with three first principles, which Newton announced in his Principia, and which are to be regarded as forming the basis of mechanical science. They may be named and defined as follows:

1. The law of *Inertia.*—A body at rest tends to remain at rest; and a body in motion tends to move forever, in a straight line, and uniformly.

That a book allows a paper to be withdrawn from under it, without itself moving, is because of its inertia. The planets continue their motion because of their inertia.

2. The law of the *Coexistence of Motions.*—A body subjected to several motions will ultimately be in the same place, whether these motions take place at the same time or successively.

A boat, under the influence of wind and tide, would be in the same place at the end of an hour, as if the wind acted alone for an hour and the tide alone for another hour. This law is fully discussed in Chapter III.

3. The law of *Action and Reaction.*—If any kind of action takes place between two bodies, it produces equal momenta in opposite directions; or, every action is accompanied by an equal and opposite reaction.

This law is illustrated by the kick of a gun. If the gun were suspended by long wires and then fired, it would be found that the momentum of the bullet would exactly equal the momentum of the gun. Owing to the difference between the weight of the bullet and of the gun, the velocity of the former is much the greater. In the collision of two railroad trains, it is immaterial as to the effects which they will respectively suffer, whether each is moving towards the other, or whether one is at rest, provided that in the latter case the moving train has a momentum equal to the momenta of the two trains in the former case. When a magnet attracts a piece of iron, each moves towards the other with the same momentum.

14. Force of Gravity.—Every mass of matter near the earth, when free to move, pursues a straight line towards its centre. The force by which this motion is produced is called *gravity*; either the gravity of the body or the gravity of the earth; for the attraction is mutual and equal, in accordance with the third law of motion. It is easy to understand why a small mass should attract a large one, as much as the large mass attracts the small one. Let A consist of *one* atom of matter, and B, at any distance from it, consist of *ten* atoms. If it be admitted that A attracts *one* atom of B as much as *that one* atom attracts A, then the above conclusion follows. For A attracts *each* of the ten atoms of B as much as *each* of the same ten attracts A; so that A exerts ten units of attraction on B, while B exerts ten units of attraction on A. The same reasoning obviously applies to the earth in relation to the small bodies on its surface.

15. Relation of Gravity and Mass.—At the same distance from the centre of the earth, *gravity varies as the mass.* This is because it operates equally on every atom of a body; hence the greater the number of atoms in a body, the greater in the same ratio is the attraction exerted upon it. That gravity varies as the mass is also proved from the observed fact, that in a vacuum it gives the same velocity, in the same time, to every mass, however great or small, and of whatever species of matter. The greater the mass, the greater the force must be to give to it the same velocity (Art. 12).

If a body is not free to move, its tendency towards the earth causes *pressure;* and the measure of this pressure is called the *weight* of the body.

Weight and mass must not be confused. The weight of a body depends upon its neighborhood to the earth. In free space it would be zero. The mass remains constant, wherever in the universe it may be. Representing the force exerted by gravity upon a body by G, and its weight by W, we have

$$G \propto m; \text{ and } w \propto m.$$

Here G represents the total force exerted by the earth upon a body. When gravity is acting upon 1 gram of matter, $G = 981$ dynes. If 2 grams were subjected to its pull, $G = 2 \times 981$ dynes.

16. Relation of Gravity and Distance.—*At different distances above the earth's surface, gravity varies inversely as the square of the distance from the centre.*

The demonstration of this proposition is reserved for astron-

omy, where it is shown by the movements of the bodies in the solar system that this law applies to them all.

The moon is 60 times as far from the earth's centre as the distance from that centre to the surface: therefore the attraction of the earth upon the particles of the moon is 3600 times less than upon particles at the surface of the earth. At the height of 4000 miles above the earth, gravity is four times less than at the surface. But the heights at which experiments are commonly made upon the weights of bodies bear so small a ratio to the radius of the earth, that this variation is commonly imperceptible. At the height of *half a mile*, the diminution does not amount to more than about $\frac{1}{4000}$th part of the weight at the surface. For, let r = the radius of the earth = 4000 miles, nearly; and let x be the height of the body, w its weight at the earth's surface, and w' its weight at the height x. Then,

$$w : w' :: (r+x)^2 : r^2 :: r^2+2rx + x^2 : r^2.$$

$$w : w - w' :: r^2+2rx+x^2 : 2rx + x^2 \therefore w - w' = \frac{w(2rx+x^2)}{r^2+2rx+x^2} \quad (A).$$

But when x is a small fraction of r, x^2 may be neglected, and the formula becomes $w - w' = \dfrac{w \times 2x}{r+2x}$ . . . . (B).

Let x be *half a mile;* then $\dfrac{w \times 1}{4000+1} = \frac{1}{4001}$th part of the whole weight; or, a body would weigh so much less at the height of half a mile than at the surface of the earth. But if the height were as great as 100 miles above the earth, the loss should be calculated by formula (A), since the other would give a result too small by one per cent. or more, according to the height.

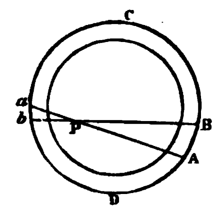
What loss of weight would a body sustain by being elevated 500 miles above the earth? *Ans.* $\frac{17}{81}$, or more than $\frac{1}{5}$ of its weight.

The relation of gravity to distance is expressed by the formula $G \propto \dfrac{1}{d^2}$; and as $G \propto m$ also, it varies as the product of the two; that is, $G \propto \dfrac{m}{d^2}$; or *gravity towards the earth varies as the mass of the body directly, and as the square of the distance from the earth's centre inversely.*

17. Gravity within a Hollow Sphere.—A particle situated *within a spherical shell* of uniform density is equally attracted in all directions, and *remains at rest.* This is true, because, in every direction from the particle, the mass varies at the same rate as the square of the distance, so that attraction increases for one reason, as much as it diminishes for the other; which is proved as follows:

Let the particle P (Fig. 1) be at any point within the spherical shell $A\ B\ C\ D$. Let two opposite cones of revolution, of very small angle, have their vertices at P, and suppose the figure to be a section through the centre of the sphere and the axis of the cones. Then $A\ B$ and $a\ b$ will be the major axes of the small ellipses, which are the bases of the cones, and which may be considered as plane figures. By geometry, $A\ P : P\ B :: P\ b : P\ a$; and the angles at P being equal, the triangles are similar; hence the angles B and a are equal. Therefore, the bases of the cones are similar ellipses, being sections of similar cones, equally inclined to the sides. By similar triangles, $\overline{A\ P}^2 : \overline{P\ b}^2 :: \overline{A\ B}^2 : \overline{a\ b}^2$. Let m and m' represent the masses of the thin laminæ which form the bases; then, since similar ellipses are to each other as the squares of their major axes, we have from the above proportion

FIG. 1.



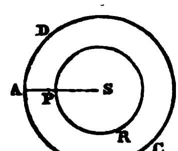
$$m : m' :: \overline{A\ P}^2 : \overline{P\ b}^2, \text{ or } \frac{m}{\overline{A\ P}^2} = \frac{m'}{\overline{P\ b}^2}.$$

But $\frac{m}{\overline{A\ P}^2}$ and $\frac{m'}{\overline{P\ b}^2}$ represent the attractions of the bases respectively on the particle (Art. 16); and since these are equal, the particle is equally attracted, but in opposite directions, by the two elliptical segments of the shell, and remains at rest. The cones of revolution may be multiplied until their bases cover the whole sphere. In each case the attractions of the opposite segments neutralize each other and the particle remains at rest.

18. Gravity within a Solid Sphere.—Within a *solid sphere* of uniform density, weight varies directly *as the distance from the centre.*

Let a particle P (Fig. 2) be within the solid sphere of $A\ D\ C$; and call its distance from the centre d. Now, by the preceding article the shell exterior to it, $A\ D\ R$, exerts no influence upon it, and it is attracted only by the sphere $P\ R\ S$. Let m represent the mass of this sphere; then gravity varies as $\frac{m}{d^2}$. But $m \propto d^3$;

FIG. 2.



$\therefore\ G \propto \frac{d^3}{d^2} \propto d$. Hence, in the earth (if it be supposed spherical and uniformly dense, though it is neither exactly), a body at the depth of 1000 miles weighs *three-fourths* as much as at the sur-

face, and at 2000 miles it weighs half as much, while at the centre it weighs nothing.

Comparing this proposition with Art. 16, we learn that just at the surface of the earth a body weighs more than at any other place without or within. Within, the weight diminishes *nearly* as the distance from the centre diminishes; without, it diminishes as the square of the distance from the centre increases.

At the surface of spheres having the same density, weight varies as the radius of the sphere. Let r be the radius of the sphere, and m its mass; then, since $G \propto \frac{m}{r^2}$, in this case it varies as $\frac{r^3}{r^2} \propto r$. Therefore, if two planets have equal densities, the weight of bodies upon them is as their radii or their diameters. If a ball two feet in diameter has the same density as the earth, a particle of dust at its surface is attracted by it nearly 21 millions of times less than it is by the earth.

19. Questions for Practice.—

1. How much weight would a rock that weighs ten tons (22,400 lbs.) at the level of the sea, lose if elevated to the top of a mountain five miles high? *Ans.* 55.8952 lbs.

2. If the earth were a hollow sphere, and if, through a hole bored through the centre, a man were let down by a rope, would the force required to support him be increased or diminished as he descended through the solid crust, and where would it become equal to nothing?

3. How much would a 44-pound shot weigh at the centre of the earth; how much at a point half-way from the centre to the surface; and how much 100 miles below the surface?

4. If a ball of the same density with the earth, $\frac{1}{10}$th of a mile in diameter, were to fall through its own diameter toward the earth, what space would the earth move through to meet the ball, the diameter of the earth being taken at 8000 miles?

Ans. $\frac{1}{80000000000}$ inch, nearly.

5. If a hole were bored through the centre of the earth, what would be the conditions of the motion of a stone dropped into the hole?

In its descent toward the centre, the force of gravity would continually decrease till at the centre it became zero; but though this force *decreases* in *intensity*, it will at each instant *increase* the previously existing velocity, though by decreasing increments, so that the stone will have its greatest velocity at the centre of the earth: it will then, in an inverse order, suffer continually *increasing* decrements of velocity until it finally comes to rest at the other surface of the earth, when it will return under similar conditions.

CHAPTER II.

VARIABLE MOTION.—WORK.—ENERGY.

20. Relation of Time and Acquired Velocity.—When a body moves with uniform motion, $s = t\,v$ (Art. 6). When a body moves with uniformly varied motion the case is somewhat different.

Let us consider the case of a body that moves with uniformly increasing velocity. Suppose the body to start from rest and at the end of the 1st second to have *acquired* a velocity of 10 cm. per second; that is to say, a velocity which would carry it over 10 cm. per second during the next and each succeeding second, if the force ceased to act at the end of the first second. Now, since the velocity is supposed to increase uniformly, we shall have at the end of the 2d second a velocity of 20 cm., at the end of the 3d a velocity of 30 cm., and so on.

Hence the first law of motion under the action of a constant force: *In uniformly accelerated motion the acquired velocities vary as the times.*

21. Space Passed Over.—Since the body started from rest and gained uniformly in velocity till it acquired a velocity of 50 cm. per second at the end of the 5th second, it is evident that its *average* velocity was 25 cm. per second; for at the start its velocity was 0 and at the end was 50; at an interval of one second after starting it had a velocity of 10, and one second before the end of the time considered it had a velocity of 40; two seconds after starting the velocity was 20, and two seconds before the end of the time the velocity was 30. Thus the less velocity at any given interval is balanced by the greater velocity during the corresponding interval of the pair, and we are thus enabled to find the distance passed over by multiplying the *average* velocity, of 25 cm. per second, by the time, 5 seconds, giving the space 125 cm.

22. Space Described during 1st Second.—We have considered the velocities at intervals of one second, but we could have chosen smaller intervals as well, and no matter how small we make our unit of time, the law holds good. Now, during the first second the body acquired a velocity of 10 cm., and if we suppose the first second to be divided into 10 equal intervals, we may apply the

same analysis to these as to the five full seconds already considered; and we find the average velocity to be $\frac{10 + 0}{2}$, or 5 cm.: hence, since the body moved for one second with a velocity which would average 5 cm. per second, it must have moved over 5 cm.: hence *a body starting from rest will, under the action of a constant force, move during the first second over a space equal to one-half the velocity acquired at the end of that second.*

23. Space Described during any Second.—The space described during *any* second is one-half the velocity impressed upon the body by the *constant force* during that second, plus the space described by reason of velocity already impressed upon the body by previous action of the force.

24. Relations of Time, Space, and Acquired Velocity.—It is necessary to know all the possible relations between the space, time, and acquired velocities. Let us now examine the relations between time and space. During the first second the body, in the case already given, moves over 5 cm. and acquires a velocity of 10 cm.; during the 2d second it will move over 10 cm. in consequence of the velocity already impressed, and over 5 cm. additional because of the continued action of the force, making a total of 15 cm. At the beginning of the 3d second the velocity is 20 cm., and the body will move over 20 cm. in consequence of this, together with 5 cm. more on account of the continued action of the force; and so on to the end of the time.

Hence, we have—

Times.	Ac. vel. at beginning of interval.	Spaces described during interval.	Total spaces.
1st sec.	0	5	For 1 sec. 5
2d "	10	15	" 2 " 20
3d "	20	25	" 3 " 45
4th "	30	35	" 4 " 80
5th "	40	45	" 5 " 125

Examining the above results, and calling the space described during the 1st second S, we have the space during

$$
\begin{aligned}
2 \text{ sec.} &= S \times 4 = S \times 2^2 \\
3 \text{ "} &= S \times 9 = S \times 3^2 \\
4 \text{ "} &= S \times 16 = S \times 4^2 \\
5 \text{ "} &= S \times 25 = S \times 5^2
\end{aligned}
$$

That is to say, *The spaces described under the action of a constant*

force are proportional to the squares of the times during which the force acts.

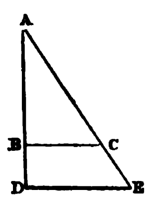
Acquired velocities are proportional to the times, and therefore the spaces must be also proportional to the squares of the acquired velocities.

25. Laws of Uniformly Accelerated Motion.—To recapitulate; when bodies move under the action of a constant force, the following relations exist between space, time, and velocity:

1. *The acquired velocities vary as the times.*
2. *The spaces vary as the squares of the times.*
3. *The spaces vary as the squares of the acquired velocities.*

As an aid to the memory, the following analogy may be employed. Let s be the space described, v the velocity acquired by a body moving from rest for the time t, s' the space described, v' the velocity acquired at any other period t'; then, from what has already been demonstrated, if t and t' be represented by the lines $A\,B$ and $A\,D$ (Fig. 3), and v and v' by the lines $B\,C$ and $D\,E$, drawn at right angles to them, s and s' will be represented by the triangles $A\,B\,C$ and $A\,D\,E$. For $A\,B\,C : A\,D\,E :: A\,B^2 : A\,D^2$; or as $B\,C^2 : D\,E^2$; or $s : s' :: t^2 : t'^2$; or as $v^2 : v'^2$. The velocity acquired varies as the time; from the similar triangles $A\,B\,C$, $A\,D\,E$, we have $B\,C : D\,E :: A\,B : A\,D$, or, $v : v' :: t : t'$.

FIG. 3.



26. Formulæ.—Let us represent by f the *acceleration due to a force*, that is to say, the increase of velocity per second due to the action of the force; then the space passed over during the 1st second, if starting from rest, would be $\frac{1}{2} f$, as deduced in Art. 22. Calling the total space s, time in seconds t, and acquired velocity v, we have, from the above laws,

$$v = ft,$$
$$s = \tfrac{1}{2} ft^2,$$
$$\text{and} \quad v^2 = 2fs.$$

27. Applications of the Formulæ.—

1. Find expressions for f in terms of v and t, s and t, v and s; for t in terms of f and s; for v in terms of s and t.

[The acceleration produced by gravity, acting for one second upon *any* freely falling body, no matter what its mass, is 981 cm., or 32.2 ft. This is generally represented by g and, when solving problems concerning falling bodies, g may be substituted for f in the formulæ of Art. 26.]

2. A body falls 10 seconds: Required (*a*) the velocity acquired; (*b*) whole distance fallen through.

Ans. (*a*) $\begin{cases} 98.1 \text{ m. per second.} \\ 322 \text{ ft. " "} \end{cases}$ (*b*) $\begin{cases} 490.5 \text{ m.} \\ 1610 \text{ ft.} \end{cases}$

3. A body has fallen through 90 meters: Required (*a*) the time of falling; (*b*) the final velocity.

4. A body falls 4 seconds and acquires a velocity of 300 feet. What was the acceleration, and what space was passed over?

Ans. $f = 75$ ft.; $s = 600$ ft.

5. A body falls through 402.5 ft.: Required (*a*) time of falling; (*b*) acquired velocity? *Ans.* (*a*) 5 sec.; (*b*) 161 ft.

28. Uniform and Uniformly Varied Motion Combined.—Thus far we have assumed the body to start from rest. If the condition be changed and the body be considered as having a uniform motion at the time the action of the constant force begins, we have merely to combine the formula for that motion with that of uniformly accelerated motion already used. Thus, if a body is thrown downward with a force which gives it a velocity of 10 m. per second, how far will it fall in 4 seconds, and what velocity will it have at the end of that time? Under the action of the downward impulse alone, it would move over 4×10 m. $= 40$ m. Under the action of gravity it would move over $4.9 \times 4^2 = 78.4$ m.; combining these two effects, we have 118.4 m. as the total distance passed over in the given time. Designating the velocity due to the impulse, usually called the "initial velocity," by v, we have total space $S = v\,t + \frac{1}{2}\,g\,t^2$; and also final velocity $v' = v + g\,t = 10 + 39.2 = 49.2$ m. per second.

29. Uniformly Retarded Motion.—In like manner we can determine the results when a constant force acts to retard velocity already imparted, by merely taking the difference of the two effects.

A body receives an impulse of 100 m. per second, and is retarded by a constant force whose acceleration is 10 m. per second; how far will the body move in 5 seconds? We now have

$$S = v\,t - \tfrac{1}{2} f\,t^2;$$
$$= 100 \times 5 - 5 \times 25 = 375.$$

30. Space in any Given Second or Seconds of Fall.—If it be required to find how far a body will move during any specified unit or units of time, proceed thus: suppose it to be required to determine how far the body will move during the 7th second; for the whole 7 seconds,

$$s = \tfrac{1}{2} f\,t^2 = \tfrac{1}{2} f \times 7^2;$$

for six seconds, $s' = \tfrac{1}{2} f\,t'^2 = \tfrac{1}{2} f \times 6^2$;

$$s - s' = \tfrac{1}{2} f\,(t^2 - t'^2) = \tfrac{1}{2} f\,(7^2 - 6^2) = \frac{13 f}{2}.$$

Suppose we are required to determine the space described during the last three seconds:

$$s = \tfrac{1}{2} f t^2 = \tfrac{1}{2} f \times 7^2;$$
$$s' = \tfrac{1}{2} f t'^2 = \tfrac{1}{2} f \times 4^2;$$
$$s - s' = \tfrac{1}{2} f (t^2 - t'^2) = \tfrac{1}{2} f (7^2 - 4^2) = \frac{33 f}{2}.$$

1. How far does a body move in the 14th second of its fall? *Ans.* 434.7 ft.

2. A body had been falling 2 minutes; how far did it move in the last second? *Ans.* 3847.9 ft.

3. What space was described in the last two seconds by a body which had fallen 300 feet? *Ans.* 214.1 ft.

4. A body had been falling $8\frac{1}{2}$ seconds; how far did it descend in the next second? *Ans.* 289.8ft.

31. Questions on Falling Bodies.—

[$g = 9.8$ m.]

1. A stone is thrown vertically upward with a velocity of 100 meters. When would it return to its original position? *Ans.* 20.4 sec.

[A little consideration will show that a body projected vertically upwards will occupy the same time in ascending as in descending. The acquired velocity upon its return is equal to the initial velocity.]

2. A stone is thrown into a pit 150 m. deep and reaches the bottom in 4 seconds: Required (*a*) initial velocity; (*b*) acquired velocity. *Ans.* (*a*) 17.9; (*b*) 57.1 m. per sec.

3. A cannon-ball has been shot vertically upward with a velocity of 250 meters in a second. After what interval of time would its velocity have been reduced to 54 meters under the retarding influence of gravity, and what space would have been traversed by the ball at the end of this time? *Ans.* 20 sec.; 3040 m.

4. A stone is thrown from a balloon with a velocity of 50 meters in a second. How soon will the velocity amount to 99 meters in a second, and through what distance will the stone have fallen? *Ans.* 5 sec.; 372.5 m.

5. How far would a body go in the 10th second of its fall?

6. A body has acquired in falling a velocity of 73.5 meters per second: Required (*a*) the time of falling; (*b*) the distance fallen through. *Ans.* (*a*) $7\frac{1}{2}$ sec.; (*b*) 275.6 m.

7. A body in falling passed over 44.1 meters in the last second: Required (*a*) the time of falling; (*b*) the distance fallen.

[$g = 32.2$ ft.]

8. An archer wishing to know the height of a tower, found that an arrow sent to the top of it occupied 8 seconds in going and returning; what was the height of the tower? *Ans.* 257.6 ft.

9. In what time would a man fall from a balloon three miles high, and what velocity would he acquire?

Ans. $t = 31.4$ sec.; $v = 1011.1$ ft.

10. A body having fallen for $3\frac{1}{2}$ seconds, was afterward observed to move with the velocity which it had acquired for $2\frac{1}{2}$ seconds more; what was the whole space described by the body?

Ans. 478.9 ft., very nearly.

11. Through what space would the aeronaut (in Question 9) fall during the last second? *Ans.* 995 ft.

12. A body has fallen from the top of a tower 340 feet high; what was the space described by it in the last three seconds?

Ans. 299.5 ft.

13. Suppose a body be projected downward with a velocity of 18 feet in a second; how far will it descend in 15 seconds?

Ans. 3892.5 ft.

14. A body is projected upward with a velocity of 65 feet in a second; how far will it rise in two seconds? *Ans.* 65.6 ft.

15. With what velocity must a stone be projected into a well 450 feet deep, that it may arrive at the bottom in four seconds?

Ans. 48.1 ft. in a second.

16. The space described in the fourth second of a fall was to the space described in the last second except four, as 1 : 3; what was the whole space described by the body? *Ans.* 3622.5 ft.

17. A staging is at the height of 84 ft. above the earth. A ball thrown upward from the earth, after an absence of 7 seconds, fell on the staging; what was the velocity of projection?

Ans. 124.7 ft. per second.

18. A body is projected upward with a velocity of 483 feet in a second; in what time will it rise to a height of 1610 feet?

Ans. $t = 3.8$ sec., or 26.2 sec.

19. From a point 386.4 feet above the earth a body is projected upward with a velocity of 161 feet in a second; in what time will it reach the surface of the earth, and with what velocity will it strike? *Ans.* $t = 12$ sec., $v = 225.4$ ft.

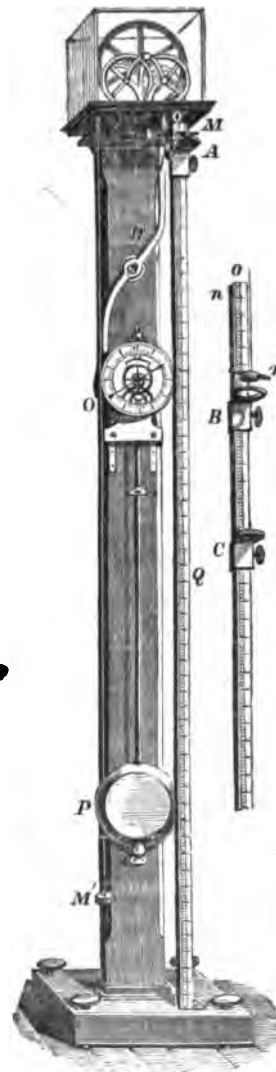
20. A body is projected upward with a velocity of 64.4 feet in a second; how far above the point of projection will it be at the end of 4 seconds? *Ans.* 0 ft.

21. A body is projected upward with a velocity of 128.8 feet in a second; where will it be at the end of 10 seconds?

Ans. 322 ft. below the point of projection.

32. Atwood's Machine.—We have seen that gravity, acting for a given time, gives the same velocity to all bodies, heavy or light. In acting upon the heavy one, however, it employs more force, and if this same force were employed in moving a still heavier body, it would give it a smaller velocity.

FIG. 4.



Now the velocities of freely falling bodies are so great that we cannot experiment upon them directly in ascertaining their laws. We can, however, make the force, exerted by gravity, on a given mass (say 1 gram), cause the motion of a heavier known mass (100 grams), and thus reduce the velocities produced, in a given ratio $(\frac{1}{100})$. The smaller velocities can then be readily observed and measured.

Employing this principle, Atwood has constructed a machine by which all the facts of uniformly accelerated or retarded motion can be illustrated with sufficient accuracy.

This machine is represented in Fig. 4. From the base of the instrument, which is furnished with leveling screws, rises a substantial pillar, about seven feet high, supporting a small table upon the top.

Above the table is a grooved wheel, delicately suspended on friction-wheels, and protected from dust by a glass case. Two equal poises, *M* and *M'*, are attached to the ends of a fine cord, which passes over the groove of the wheel. As gravity exerts equal forces on *M* and *M'*, they are in equilibrium. To set them in motion, a small bar, *m*, is placed on *M*, which will immediately begin to descend, and *M'* to rise. But this motion will be slower than in falling freely, because the force which gravity exerts on the bar must be communicated to the poises, and also to the revolving wheel over which the cord passes. By increasing the poises *M*, *M'*, and diminishing the bar *m*, the motion may be

made as slow as we please. O is a simple clock attached to the pillar for measuring seconds, and for dropping the poise M at the beginning of a vibration of the pendulum. Q is a scale of centimeters or inches extending from the base to the table. The stage A may be clamped to any part of the scale, in order to stop the poise M in its descent, as represented at C. The ring B, which is large enough to allow the poise, but not the bar, to pass through it, is also clamped to the scale wherever the acceleration is to cease.

Let M be raised to the top, and held in place by a support, and then let the pendulum be set vibrating. When the index passes the zero point the clock causes the support to drop away, and the poise descends. The pendulum shows how many seconds elapse before the bar is arrested by the ring, and how many more before the poise strikes the stage. From the top to the ring the motion is accelerated by the constant fraction of gravity acting on it; from the ring to the stage the poise moves uniformly with the acquired velocity. Moreover, the resistance of the air is so much diminished when the motion is slow, that a good degree of correspondence is found to exist between the experiments and the results of calculation.

If we disregard the mass of the wheel as not sensibly affecting the results, which we may do in practice if the weights are heavy as compared with it, we may illustrate the action of the machine by the following case: Suppose M and M' to weigh together 99 grams and m to weigh 1 gram. The force exerted on 1 gram must move 100 grams, and hence the velocity will be but $\frac{1}{100}$ as great as by free falling. The acquired velocity would then be but 9.8 cm. at the end of the first second, and would pass over but 4.9 cm. during the first second. Such a small velocity can be readily observed.

Ex. 1. If $M = M' = 24.5$ grams, and $m = 1$ gram what is the acceleration?

33. Work.—*Work is the production of motion against resistance.* Whenever a body is set in motion against the restraining influence of any force, work is said to be performed. If no motion be produced, then no work is performed. If there be no opposing force, no work is performed.

Whenever a weight is lifted from the ground, work is done against gravity; the same is true when a hill or pair of stairs are ascended. Winding up a watch requires work against the force of elasticity. Turning a grindstone demands work against friction.

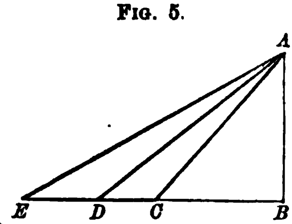
Let A represent the work done, F the force performing the work, and s the distance moved in the direction of this force, then work is calculated by the formula,

$$A = Fs.$$

In order to get a unit for work we must make F and s each equal to unity. This unit of work, called *The Erg, is the work performed by the force* 1 *dyne in moving a body* 1 *centimeter in the direction it is acting.*

This unit is very small (the force exerted by gravity on one gram = 981 dynes), and hence the *Megalerg* (= 1,000,000 ergs) is used. The practical unit, which is still much used, the *Foot-pound*, takes the force exerted by gravity on a pound of matter for F, and the foot for unit s. It is often convenient, in measuring work, to consider the work done as equal to the product of the force which opposes motion, and the motion produced resolved in the direction of this opposing force. This is especially the case when gravity is the opposing force. Thus the force used in raising a weight is in a vertically upward direction—the resisting force, gravity, working vertically downward. The work performed equals the pull of gravity times the *vertical* distance moved through. No more work would be performed in climbing a ladder to the top of a precipice than by climbing the hill to the rear of it. Thus if gravity be supposed to work in the direction $A\ B$ (Fig. 5), the same amount of work will be performed in rising to A, by each of the paths $B\ A$, $C\ A$, $D\ A$, and $E\ A$.

FIG. 5.



In practical life, though the hod-carrier and telegraph-pole climber would appear to have much work to perform, the ditch digger is said to do the most work in a day.

Ex. 1. How much work is performed in raising a liter of water 1 m? *Ans.* 98.1 megalergs.

Ex. 2. A plank 5 feet long reaches from the threshold of a warehouse door to a plate on the ground 4 feet from the building. What work is performed in rolling a $333\frac{1}{3}$ pounds cask from the ground into the warehouse? *Ans.* 1000 foot-pounds.

34. Energy.—*Energy is the capacity to do work.* The energy possessed by bodies is divided into two classes.

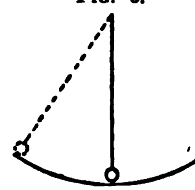
1. *Potential energy is the energy which a body has in virtue of its position.* The weights of a clock can do work in virtue of their position relative to the earth. Water at a high level also has potential energy. A wound-up spring has energy from the strained position of its molecules.

2. *Kinetic energy is the energy which a body has in virtue of its motion.* A rifle-bullet because of its velocity has energy, and by proper mechanism could be made to raise a weight. In fact, if

fired vertically upward, it not only does work in lifting itself, but, if the distance it rises and its weight are known, we can measure the work it does.

35. Transmutation of Energy.—The rifle-bullet, in its ascent, gradually loses its kinetic energy until, at the highest point, it possesses none. What has become of it? Having lost its motion it now possesses potential energy because of its position relative to the earth. Falling backward it *transmutes* its potential into kinetic energy until, arriving at the starting-point, it has no potential but all kinetic energy. During all the time of its flight its total energy, kinetic plus potential, was constant, and it returns with the same velocity that it started with. A swinging pendulum also illustrates the transmutation of energy. At either extremity of its swing (Fig. 6), its energy is all potential. At the lowest point of the arc it is all kinetic. At intermediate points it possesses both kinds.

FIG. 6.



36. Calculation of Energy.—Potential energy is possessed by a body because work has been performed upon it. It is capable of performing the same amount of work by returning to its former position. To calculate its potential energy we have only to calculate the work necessary to bring it to its position. A mass m, raised vertically through s cm., requires a force of $m\,g$ dynes, and the work performed, that is, its

$$\textit{Potential energy} = m\,g\,s, \text{ ergs.}$$

Thus 2 grams raised 1 m. have a potential energy $= 2 \times 981 \times 100 = 186{,}200$.

In the case of the kinetic energy of a mass m, moving with a velocity v, we must know how much work it can do on itself, or how high it can raise itself vertically. From Art. 26 we know that this distance,

$$s = \frac{v^2}{2g}.$$

Multiplying both sides by the acting force $m\,g$, we have

$$m\,g\,s = \tfrac{1}{2}\,m\,v^2 = \textit{kinetic energy.}$$

All the kinetic energy has been transmuted into potential, which is measured by $m\,g\,s$.

It is often required to calculate the kinetic energy in foot-pounds. To do this we have to remember that weight,

$$W = m\,g.$$

Substituting this value for $m\,g$ in the formula above we have

$$W\,s = \tfrac{1}{2}\,\frac{(m\,g)}{g}\,v^2 = \frac{W\,v^2}{2g}.$$

If W is measured in pounds, and v and g in feet, the value obtained is in foot-pounds.

37. Conservation and Dissipation of Energy.—One of the most fundamental and important principles in the whole subject of physics is the law of the conservation of energy. It may be stated as

The total amount of energy in the universe is constant: No energy is ever lost.

Whenever a certain amount of energy is communicated to machinery by an engine, the full amount is not recovered in what the machinery does. The rest is said to be lost. Tracing this loss we find a great deal owing to friction. The overcoming of friction, however, produces heat, another form of energy. This heat eventually escapes to the earth, and there may be of use in furthering vegetable life. Vegetables are possessed of potential energy, and by their assimilation in the human system, and combination with the oxygen of the air, reappear as animal heat or muscular energy. If we trace out the course of a certain amount of energy, we will find that it is continually changing itself from one form into another, and eventually turns into heat. Tracing its past history, we find that by far the larger proportion came ultimately from the sun. Thus water in a mill-pond owes its potential energy to the evaporating influence of the sun. The winds are caused by the sun's heat, but the tides largely by the moon's attraction.

When a lead bullet is fired against a stone wall its energy is transformed into heat. This may melt the lead, but finally the heat goes into the surrounding objects and is at a low temperature. Heat at low temperature is of no use to man in performing work, and hence the energy which it represents is said to be *dissipated*. Now, though the total energy of the universe remains constant, the amount which is available to man is decreasing. Clausius, considering the ultimate form of energy to be heat, called the expression,

$$\frac{\text{total quantity of heat}}{\text{average temperature}} = \text{entropy},$$

and expressed the above fact by saying, "The entropy of the universe approaches a maximum."

38. Power.—*Power is the rate of performance of work.* If t represents the time required to perform an amount of work A, then the power

$$P = \frac{A}{t}.$$

There are two practical units of power, represented by the following equations:

$$\text{Watt} = \frac{10^7 \text{ ergs}}{\text{per second}}.$$

$$\text{Horse-power} = \frac{33000 \text{ ft.-lbs.}}{\text{per minute}}.$$

One horse-power is equivalent to 746 Watts.

39. Problems on Energy.—

1. What is the potential energy of 25 kilos. raised to a height of 40 meters? *Ans.* 981×10^8 ergs.

2. A stone, weighing 6 kilos., falls from rest, at a place where $g = 980$; what will be its kinetic energy at the end of 5 seconds? *Ans.* 7.203×10^{10} ergs.

3. A stone weighing 10 grams, fired vertically upward, returns in 10 seconds; when was its kinetic energy greatest, and how much? When did it have maximum potential energy, and how much?

4. Find the horse-power of an engine that should be employed for raising coal from a mine 200 feet deep, the average daily (24 hours) yield being 1782 tons. *Ans.* 16.8 horse-power.

5. A man weighing 75 kilos. can climb by stairs to a floor 20 meters above in 1 minute. What is his power?

CHAPTER III.

COMPOSITION AND RESOLUTION OF MOTION.

40. Motion by Two or More Forces.—Motion produced by a single force, either impulsive or continued, has been already considered. But motion is more generally caused by several forces acting in different directions.

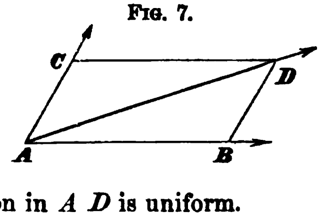
When two or more forces act at once on a body, each force is called a *component*, and the joint effect is called the *resultant*. Forces may be represented by the *straight lines* along which they would move a body in a given time; the lines represent the forces in two particulars, the *directions* in which they act and their *relative magnitudes*. Whenever an arrow-head is placed on a line, it shows in which of the two directions along that line the force acts.

41. The Parallelogram of Forces.—This is the name given to the relation which exists between any two components and their resultant, and is stated as follows:

If two forces acting at once on a body are represented by the adjacent sides of a parallelogram, their resultant is expressed by the diagonal which passes through the intersection of those sides.

Suppose that a body situated at A (Fig. 7) receives an impulse which, acting alone, would carry it over $A\ B$ in a given time, and another which would carry it over $A\ C$ in the same length of time. If both impulses are given at the same instant, the body describes $A\ D$ in the same time as $A\ B$ by the first force, or $A\ C$ by the second, and the motion in $A\ D$ is uniform.

FIG. 7.



This is an instance of the coexistence of motions, stated in the second law of motion (Art. 13). For the body, in passing directly from A to D, is making progress in the direction $A\ C$ as rapidly as though the force $A\ B$ did not exist; and at the same time it advances in the direction $A\ B$ as fast as though that were the

only force. When the body reaches D, it is as far from the line $A\ B$ as if it had passed over $A\ C$; it is also as far from the line $A\ C$ as if it had gone over $A\ B$. Thus it appears that both motions $A\ B$ and $A\ C$ fully coexist in the progress of the body along the diagonal $A\ D$. That the motion is uniform in the diagonal is evident from the law of inertia; for the body is not acted on after it leaves A.

It is evident that a single force might produce the same effect; that force would be represented, both in direction and magnitude, by the line $A\ D$. The force $A\ D$ is said to be equivalent to the two forces $A\ B$ and $A\ C$.

42. Velocities Represented.—The lines $A\ B$ and $A\ C$ are described by the components separately, and the line $A\ D$ by their joint action, *in the same length of time.* Hence the *velocities* in those lines are as the lines themselves. In the parallelogram of forces, therefore, two adjacent sides and the diagonal between them represent—

1st. The relative *directions* of the components and resultant;

2d. Their relative *magnitudes;* and

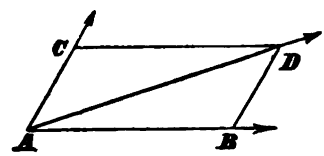
3d. The relative *velocities* with which the lines are described.

43. The Triangle of Forces.—For purposes of calculation, it is more convenient to represent two components and their resultant by the sides of a triangle, than by the sides and diagonal of a parallelogram. In Fig. 7, $C\ D$, which is equal and parallel to $A\ B$, may represent in direction and magnitude the same force which $A\ B$ represents. Therefore, the components are $A\ C$ and $C\ D$, while their resultant is $A\ D$; and the angle C in the triangle is the supplement of $C\ A\ B$, the angle between the components. Care should be taken to construct the triangle so that the sides representing the components may be taken in succession in the directions of the forces, as $A\ C$, $C\ D$; then $A\ D$ correctly represents their resultant. But, although $A\ C$ and $A\ B$ represent the components, the third side, $C\ B$, of the triangle $A\ C\ B$, does not represent their resultant, since $A\ C$ and $A\ B$ cannot be taken successively in the direction of the forces. It is necessary to go back to A in order to trace the line $A\ B$. It should be observed, that though $C\ D$ represents the *magnitude* and *direction* of the component, it is not in the *line* of its action, because both forces act through the same point A.

Three forces produce equilibrium when they may be represented in direction and magnitude by the three sides of a triangle taken in order.

For, when three forces are in equilibrium, one of them must be equal to, and opposite to, the resultant of both the others. But the forces $A\ C$ and $A\ B$ (Fig. 8) produce the resultant $A\ D$; therefore the equal and opposite force $D\ A$, since it is in equilibrium with $A\ D$, is also in equilibrium with $A\ C$ and $A\ B$, or $A\ C$ and $C\ D$. Hence the three forces $A\ C$, $C\ D$, and $D\ A$, taken in order around the figure, produce equilibrium.

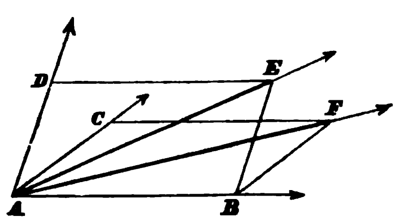
Fig. 8.



44. The Forces Represented Trigonometrically.—Since the sides of a triangle are proportional to the sines of the opposite angles, these sines may also represent two components and their resultant. Thus, the sine of $C\ A\ D$ corresponds to the component $A\ B$ ($= C\ D$); the sine of $C\ D\ A$ ($= D\ A\ B$) corresponds to the component $A\ C$; and the sine of C ($=$ sine of $C\ A\ B$) corresponds to the resultant $A\ D$. Each of the three forces is therefore represented by the sine of the angle between the other two.

45. Greatest and Least Values of the Resultant.—A change in the angle between the components alters the value of the resultant; as the angle increases from 0° to 180°, the resultant diminishes from the *sum* of the components to their *difference*. In Fig. 9, let $C\ A\ B$ and $D\ A\ B$ be two different angles between the same components $A\ C$ (or $A\ D$) and $A\ B$. As $C\ A\ B$ is less than $D\ A\ B$, its supplement $A\ B\ F$ is greater than $A\ B\ E$, the supplement of $D\ A\ B$; therefore $A\ F$ is greater than $A\ E$. When the angle $C\ A\ B$ is diminished to 0°, the sides $A\ B$, $B\ F$, become one straight line, and $A\ F$ equals their *sum;* when $D\ A\ B$ is enlarged to 180°, E falls on $A\ B$, and $A\ E$ equals the *difference* of $A\ B$ and $A\ C$. Between the sum and difference of the components, the resultant may have all possible values.

Fig. 9.



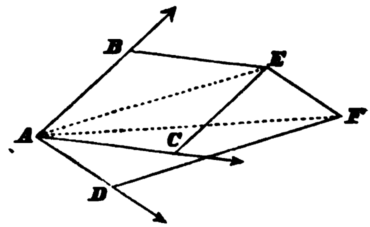
Two forces produce equilibrium when they are equal and act upon the same point in opposite directions.

Since two forces produce the least resultant when they act at an angle of 180° with each other, and the resultant then equals the *difference* of the forces, if the forces are equal, their difference

is zero, and the resultant vanishes; that is, the two forces produce equilibrium.

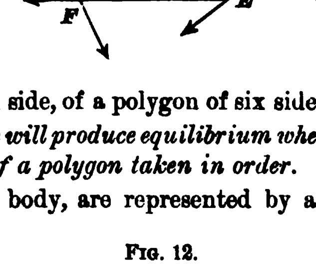
46. The Polygon of Forces.—All the sides of a polygon except one may represent so many forces acting at the same time on a body, and the remaining side will represent their resultant. In Fig. 10, suppose $A\ B$, $A\ C$, and $A\ D$, to represent three forces acting together on a body at A. The resultant of $A\ B$ and $A\ C$ is represented by the diagonal $A\ E$; and the resultant of $A\ E$ and $A\ D$ by the diagonal $A\ F$. As $A\ F$ is equivalent to $A\ E$ and $A\ D$, and $A\ E$ is equivalent to $A\ B$ and $A\ C$, therefore $A\ F$ is equivalent to the three, $A\ B$, $A\ C$, and $A\ D$. But if we substitute $B\ E$ for $A\ C$, and $E\ F$ for $A\ D$, then the three components are $A\ B$, $B\ E$, and $E\ F$, three sides of a polygon, and the resultant $A\ F$ is the fourth side of the same polygon.

Fig. 10.



So, in Fig. 11, $A\ B$, $B\ C$, $C\ D$, $D\ E$, and $E\ F$, may represent the directions and relative magnitudes of five forces, which act simultaneously on a body at A. The resultant of $A\ B$ and $B\ C$ is $A\ C$; the resultant of $A\ C$ and $C\ D$ is $A\ D$; the resultant of $A\ D$ and $D\ E$ is $A\ E$; and the resultant of $A\ E$ and $E\ F$ is $A\ F$; which last is therefore the resultant of all the forces, $A\ B$, $B\ C$, $C\ D$, $D\ E$, and $E\ F$; the components being represented by five sides, and their resultant by the sixth side, of a polygon of six sides.

Fig. 11.



More than three forces in one plane will produce equilibrium when they can be represented by the sides of a polygon taken in order.

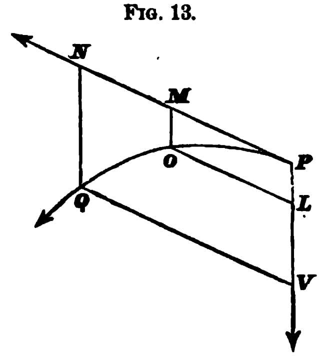
Since several forces acting on a body, are represented by all the sides of a polygon except one, their resultant is represented by the remaining side. Thus, the resultant of the forces $A\ B$, $B\ C$, $C\ D$, and $D\ E$ (Fig. 12), is $A\ E$. Now, the force $E\ A$, equal and opposite to $A\ E$, since it would be in equilibrium with $A\ E$, is therefore in equilibrium with all the

Fig. 12.

others. Hence the forces $A\,B$, $B\,C$, $C\,D$, $D\,E$, and $E\,A$, taken in order around the figure, are in equilibrium.

47. Curvilinear Motion.—Since, according to the first law of motion, a moving body proceeds in a straight line, if no force disturbs it, whenever we find a body describing a *curve*, it is certain that some force is continually deflecting it from a straight line. Besides the original impulse, therefore, which gave it motion in one direction, it is subject to the action of a continued force, which operates in another direction. A familiar example occurs in the path of a projectile. Suppose a body to be thrown from P (Fig. 13), with an impulse which would alone carry it to N, in the same time in which gravity alone would carry it to V. Complete the parallelogram $P\,Q$; then, as both motions coexist (2d law), the body at the end of the time will be found at Q. Let t be the time of describing $P\,N$ or $P\,V$; and let t' be the time of describing $P\,M$ by the impulse, or $P\,L$ by gravity. Then, at the end of the time t', the body will be at O. Now, as $P\,N$ is described uniformly, $P\,N : P\,M :: t : t'$; $\therefore P\,N^2 : P\,M^2 :: t^2 : t'^2$.

FIG. 13.



But (Art. 25), $P\,V : P\,L :: t^2 : t'^2$; $\therefore P\,V : P\,L :: P\,N^2 : P\,M^2$; or $Q\,V^2 : O\,L^2$.

Hence, the curve is such that $P\,V \propto Q\,V^2$; that is, the abscissa varies as the square of the ordinate, which is a property of the parabola. $P\,O\,Q$ is therefore a parabola, one of whose diameters is $P\,V$, and the parameter to that diameter is $\frac{Q\,V^2}{P\,V}$.

Owing to the resistance of the air, the curve deviates sensibly from a parabola, especially in swift motions.

48. Calculation of the Resultant of Two Impulsive Forces.—When two components and the angle between them are given, the resultant may be found both in direction and magnitude by trigonometry. The theorem required is that for solving a triangle, when two sides and the included angle are given; but the included angle is not that between the components, but its supplement (Art. 43). In Fig. 7, if $A\,B = 54$, and $A\,C = 22$, and $C\,A\,B = 75°$, then $A\,C\,D$ is the triangle for solution, in which $A\,C = 22$, $C\,D = 54$, and $A\,C\,D = 105°$. Performing the calculation, we find the resultant $A\,D = 63.363$, and the angle

$D\,A\,B$, which it makes with the greater force, $= 19°\ 35'\ 43''$. This method will apply in all cases.

1. A foot-ball received two blows at the same instant, one directly east at the rate of 71 feet per second, the other exactly northwest, at the rate of 48 feet per second; in what direction and with what velocity did it move?

Ans. N. 47° 30′ 52″ E. Vel. =50.253.

The process is of course abridged, if the forces act at a right angle with each other, as in the following example:

2. A balloon rises 1120 feet in one minute, and in the same time is borne by the wind 370 feet; what angle does its path make with the vertical, and what is its velocity per second?

Ans. 18° 16′ 53″; $v = 19.659$.

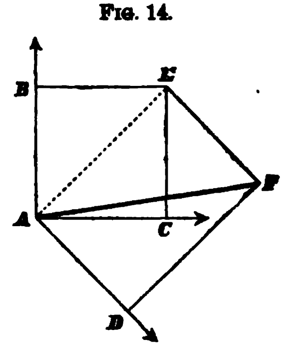
In the next example, one component and the angle which each component makes with the resultant, are given to find the resultant and the other component.

3. From an island in the Straits of Sunda, we sailed S. E. by S. (33° 45′) at the rate of 6 miles an hour; and being carried by a current, which was running toward the S. W. (making an angle with the meridian of 64° 12¼′), at the end of four hours we came to anchor on the coast of Java, and found the said island bearing due north; required *the length of the line* actually described by the ship, and *the velocity of the current?*

Ans. $s = 26.4$ miles.
$v = 3.7024$ miles per hour.

If the magnitudes and directions of any number of forces are given, the resultant of them all is obtained by a repetition of the same process as for two. In Fig. 11, first calculate $A\,C$, and the angle $A\,C\,B$, by means of $A\,B$, $B\,C$, and the angle B. Subtracting $A\,C\,B$ from $B\,C\,D$, we have the same data in the next triangle, to calculate $A\,D$, and thus proceed to the final resultant, $A\,F$.

FIG. 14.



As it is immaterial in what order the components are introduced into the calculation, it will diminish labor, to find first the resultant of any two *equal* components, or any two which make a *right angle* with each other; since it can be done by the solution of an isosceles, or a right-angled triangle.

4. The particle A (Fig. 14) is urged by three equal forces $A\,B$, $A\,C$, and $A\,D$; the angle $B\,A\,C = 90°$, and $C\,A\,D = 45°$;

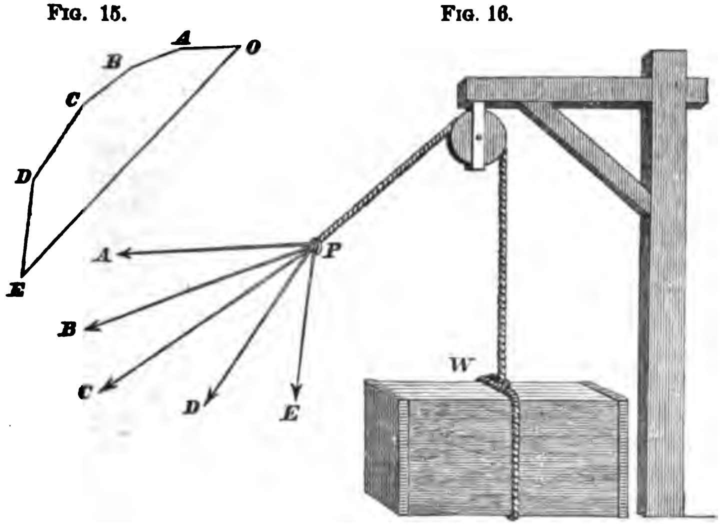
what is the direction of the resultant, and how many times $A B$?

Ans. $B A F = 80° 16'$, and $A F : A B :: \sqrt{3} : 1$.

5. Five sailors raise a weight by means of five separate ropes, in the same plane, connected with the main rope that is fastened to the weight in the manner represented in Fig. 16. B pulls at an angle with A of 20°; C with B, 19°; D with C, 21° 30′; and E with D, 25°. A, B, and C, pull with equal forces, and D and E with forces one-half greater; required the magnitude and direction of the resultant.

Ans. Its angle with A is 46° 33′ 10″. Its magnitude is 5.1957 times the force of A.

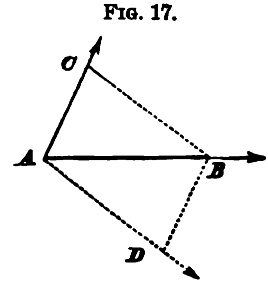
FIG. 15. FIG. 16.



If the polygon $O A B C D E$ (Fig. 15) be constructed for the above case, $O B$ and $C E$ are easily calculated in the isosceles triangles $O A B$ and $C D E$, after which $O C$ and then $O E$ are to be obtained by the general theorem.

49. The Resultant and all Components, except one, being given, to Find that one Component.—If $A B$ (Fig. 17) is the resultant to be produced, and there already exists the force $A C$, a second force can be found, which acting jointly with $A C$, will produce the motion required. Join $C B$, and draw $A D$ equal and parallel to it, then $A D$ is the force re-

FIG. 17.



quired; for $A\,B$ is equivalent to $A\,C$ and $C\,B$. Therefore $C\,B$ has the magnitude and direction of the required force; $A\,D$ is the line in which it must act.

Again, suppose that *several* forces act on A, and it is required to find the force which, in conjunction with them all, shall produce the resultant $A\,B$. Let the several forces be combined into one resultant, and let $A\,C$ represent that resultant. Then $A\,D$ may be found as before.

The trigonometrical process for finding a component is essentially the same as for finding a resultant.

1. A ferry-boat crosses a river $\frac{3}{4}$ of a mile broad in 45 minutes, the current running all the way at the rate of 3 miles an hour; at what angle with the direct course must the boat head up the stream in order to move perpendicularly across? *Ans.* 71° 34′.

2. A sloop is bound from the mainland of Africa to an island bearing W. by N. (78° 45′) distant 76 miles, a current setting N. N. W. (22° 30′) 3 miles an hour; what is the *course* to arrive at the island in the shortest time, supposing the sloop to sail at the rate of 6 miles per hour; and what *time* will she take?

Ans. Course, S. 76° 41′ 4″ W. Time, 10 h. 40 m. 7 sec.

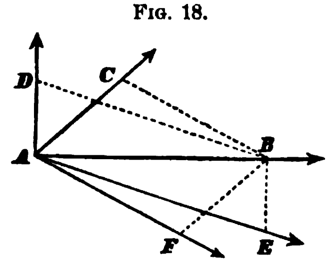
3. The resultant of two forces is 10; one of them is 8, and the direction of the other is inclined to the resultant at an angle of 36°. Find the angle between the two forces.

Ans. 47° 17′ 5″ or 132° 42′ 55″.

4. A ball receives two impulses: one of which would carry it N. 27 feet per second; the other N. 60° E. with the same velocity; what third impulse must be conjoined with them, to make the ball go E. with a velocity of 21 ft? *Ans.* S. 3° 22′ W. $v = 40.57$.

50. Resolution of Motion.—In the *composition* of motions or forces, the resultant of any given components is found; in the *resolution* of motion or force, the process is reversed; the resultant being given, the components are found, which are equivalent to that resultant.

FIG. 18.



If it be required to find what two components can produce the resultant $A\,B$ (Fig. 18), we have only to construct on $A\,B$, as a base, any triangle whatever, as $A\,B\,C$ or $A\,B\,D$ (Art. 43); then, if $A\,C$ is one component, the other is $A\,F$, equal and parallel to $C\,B$; or if $A\,D$ is one, the other is $A\,E$, equal and parallel to $D\,B$; and so for any triangle whatever on the base $A\,B$. The number of pairs is therefore infinite, whose resultant in each case is $A\,B$.

The *directions* of the components may be chosen at pleasure, provided the sum of the angles made with $A\ B$ is less than two right angles.

The *magnitude* and *direction* of one component may be fixed at pleasure.

The *magnitudes* of both components may be what we please, provided their difference is not greater, and their sum not less, than the given resultant.

These conditions are obvious from the properties of the triangle.

When a given force has been resolved into two others, each of those may again be resolved into two, each of those into two others still, and so on. Hence it appears that a given force may be resolved into any number of components whatever, with such limitations as to direction and magnitude as accord with the foregoing statements.

1. A motion of 153 toward the north is produced by the forces 100 and 125; how are they inclined to the meridian?

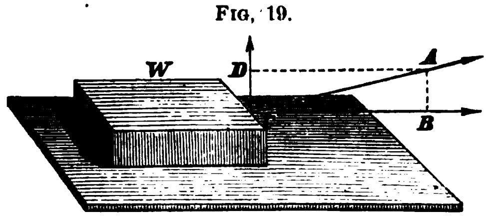
Ans. 54° 28′ and 40° 37′ 7″.

2. A resultant of 617 divides the angle between its components into 28° and 74°; what are the components?

Ans. 606.34 and 296.14.

51. Resolution of a Force, to Find its Efficiency in a Given Direction.—By the resolution of a force into two others acting at right angles with each other, it is ascertained how much efficiency it exerts to produce motion in any given direction. For example, a weight W (Fig. 19), lying on a hori-

FIG. 19.



zontal plane, and pulled by the oblique force $C\ A$, is prevented by gravity from moving in the line $C\ A$, and is compelled to remain on the plane. Resolve $C\ A$ into $C\ B$, *in* the plane, and $C\ D$ *perpendicular* to it: then the former represents the component which is efficient to cause motion along the plane; the latter has no influence to aid or hinder that motion; it simply diminishes pressure upon the plane. In like manner, if $A\ C$ is an oblique force, *pushing* the weight, its horizontal component, $B\ C$,

is alone efficient to move it; the other, $A\ B$, merely increasing the pressure. Representing by F the whole force, by f and f' the components in direction of motion and resistance respectively, and by a the angle of inclination, we have

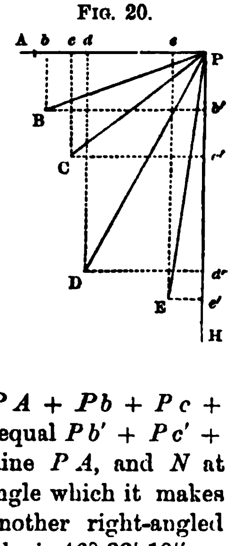
$$f = F \cos a$$
$$\text{and} \quad f' = F \sin a.$$

If only 88 per cent. of the strength of a horse is efficient in moving a boat along a canal, what angle does the rope make with the line of the tow-path? *Ans.* 28° 21′ 27″.

52. Resultant found by means of Rectangular Axes.—When several forces act in one plane upon a body, their resultant may be conveniently found by the use of right-angled triangles alone. Select at pleasure two lines at right angles to each other, both of them lying in the plane of the forces, and passing through the point at which the forces are applied. These lines are called *axes*. The following example illustrates their use:

Let $P\ A$, $P\ B$, $P\ C$, $P\ D$, $P\ E$ (Fig. 20) represent the forces in Question 5 (Art. 48). Let one axis, for convenience, be chosen in the direction $P\ A$, and let $P\ H$ be drawn at right angles to it for the other axis. These axes are supposed to be of indefinite length. Then proceed as in Art. 51 to resolve each force into two components on these axes. As $P\ A$ acts in the direction of one axis, it does not need to be resolved. The Projections of $P\ B$ are

FIG. 20.



$$P\,b = P\,B \times \cos 20°$$
$$P\,b' = P\,B \times \sin 20°;$$

again,

$$P\,c = P\,C \times \cos 39°$$
$$P\,c' = P\,C \times \sin 39°, \text{ etc.}$$

Suppose $P\ A$ produced so as to equal $P\,A + P\,b + P\,c + P\,d + P\,e = M$, and $P\ H$ produced so as to equal $P\,b' + P\,c' + P\,d' + P\,e' = N$. Now, as M acts in the line $P\ A$, and N at right angles to it, their resultant and the angle which it makes with $P\ A$ are found by the solution of another right-angled triangle. The resultant is 5.1957, and the angle is 46° 33′ 10″, as in Art. 48.

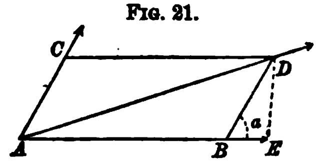
If any components of the resolved forces are opposite to $P\ A$ or $P\ H$, they are reckoned as negative quantities.

53. Analytical Expression for the Resultant.—Put $A\ C$ (Fig. 21) $= P$, $A\ B = P'$, $A\ D = R$, angle $C\ A\ B = a$; then in triangle $A\ B\ D$ we have, by Geometry, $\overline{A\ D}^2 = \overline{A\ B}^2 + \overline{B\ D}^2 +$

$2\,AB \times BE$, but $BE = BD\cos a = P\cos a$, and hence substituting as above $R^2 = P'^2 + P^2 + 2\,P'P\cos a$; whence

$$R = \sqrt{P'^2 + P^2 + 2\,P'P\cos a}\;. \quad (1.)$$

Fig. 21.



Hence, *The resultant of any two forces, acting at the same point, is equal to the square root of the sum of the squares of the two forces, plus twice the product of the forces into the cosine of the included angle.*

If $a = 0$, its cosine will be 1, and (1) becomes

$$R = P + P'.$$

If $a = 90°$, its cosine will be 0, and we shall have

$$R = \sqrt{P^2 + P'^2}.$$

If $a = 180°$, its cosine will be -1, and we shall have

$$R = P - P'.$$

1. Two forces, P and P', are equal in intensity to 24 and 30, respectively, and the angle between them is 105°; what is the intensity of their resultant? *Ans.* 33.21.

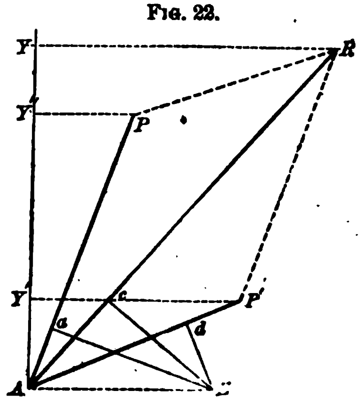
2. Two forces, P and P', whose intensities are, respectively, equal to 5 and 12, have a resultant whose intensity is 13; required the angle between them. *An.* 90°.

3. A boat is impelled by the current at the rate of 4 miles per hour, and by the wind at the rate of 7 miles per hour; what will be her rate per hour when the direction of the wind makes an angle of 45° with that of the current? *Ans.* 10.2 miles.

4. Two forces and their resultant are all equal; what is the value of the angle between the two forces? *Ans.* 120°.

54. Principle of Moments.—The *moment* of a force, with respect to a point, is the product of the force into the perpendicular let fall from the point to the line of direction of the force.

Fig. 22.



The fixed point is called the centre of *moments;* the perpendicular distance, the *lever-arm of the force;* and the *moment* measures the tendency of the force to produce rotation about the centre of moments.

Denote the forces (Fig. 22) by P, P' and their resultant by R. From E any point in the

plane of the forces let fall upon the directions of the forces the perpendiculars $E\,a$, $E\,c$, $E\,d$. Represent these by l, L, l'. Draw two rectangular axes of reference as in Art. 52, so that one of them may pass through A and E. The projection of the resultant R is equal to the sum of the projections of its components (Art. 52); hence,

$$A\,Y = A\,Y'' + A\,Y' \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By similar triangles $A\,c\,E$ and $A\,R\,Y$, we have

$$A\,Y : E\,C = L :: A\,R = R : A\,E \quad \therefore A\,Y = \frac{L \times R}{A\,E};$$

by similar triangles $A\,a\,E$ and $A\,P\,Y''$, we have

$$A\,Y'' : E\,a = l :: A\,P = P : A\,E \quad \therefore A\,Y'' = \frac{l \times P}{A\,E};$$

and by similar triangles $A\,d\,E$ and $A\,P'\,Y'$, we have

$$A\,Y' : E\,d = l' :: A\,P' = P' : A\,E \quad \therefore A\,Y' = \frac{l' \times P'}{A\,E};$$

substituting these values in Eq. (1) and multiplying by $A\,E$, we have

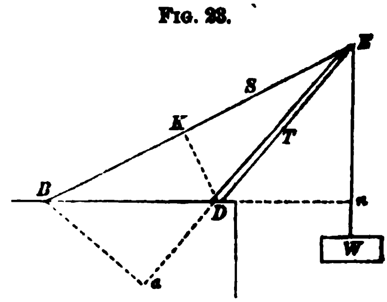
$$R \times L = P \times l + P' \times l'.$$

Hence, *the moment of the resultant of two forces with respect to any point is equal to the algebraic sum of the moments of the forces taken separately.*

By using the resultant as above and a third force the moment of the resultant of the three forces may be proved equal to the algebraic sum of the moments of the forces, and so on for any number of forces.

To illustrate the application of the principle of moments, suppose a weight W of 1000 lbs. to be suspended from the end of a spar $D\,E$ as in the figure; required the strain upon the stay $E\,B$.

FIG. 28.



We have three forces in equilibrium acting at E, viz., the weight W, the strain upon the rope S, and the upward thrust of the spar T. If we select the centre of moments upon the line of either force, the moment of that force will be zero. As the thrust T is equal and opposed to the resultant of S and W, we will take D as the centre of moments, and we have,

Moment of $T =$ moment of $S +$ moment of W; but $T \times 0 =$ moment of T, $S \times K\,D =$ moment of S, and $W \times D\,n =$ mo-

ment of W. But W tends to cause E to revolve towards the right about the point D, while S tends to cause revolution of E towards the left; hence one must be regarded as a positive and the other as a negative moment, and we have, finally, if we call $W \times Dn$ positive, $0 = -S \times KD + W \times nD$, whence, $S = \frac{W \times nD}{KD}$.

If in the problem, $DE = 20$ ft, $BD = 20$ ft., and $EBD = 30°$, then will $KD = BD$ sine $30° = 10$, and $Dn = DE$ sine $30° = 10$.

$$\therefore S = \frac{1000 \times 10}{10} = 1000 \text{ lbs.}$$

To find the thrust at D, take the point B as a centre of moments, then

$$T \times Ba = S \times 0 + W \times Bn\,;\ \therefore T = \frac{W \times Bn}{Ba}.$$

Now $Bn = BD + Dn = 30$; to find Ba, we have

$$KE = \sqrt{\overline{DE}^2 - \overline{KD}^2} = \sqrt{300}, \text{ and } BE = 2\,KE = 2\sqrt{300}.$$

In similar triangles KDE and BEa we have

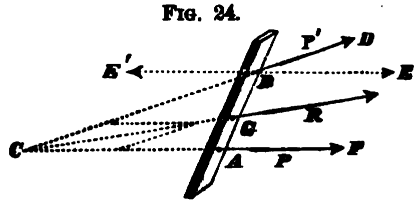
$$DE : BE :: KD : Ba\,;$$

$$\therefore Ba = \frac{2\sqrt{300} \times 10}{20} = \sqrt{300}\,;\ \text{hence } T = \frac{1000 \times 30}{\sqrt{300}}.$$

Remember that when *three* forces, acting at the same point, are in equilibrium, one of the three being known, either of the other two can be found by taking the centre of moments on the line of the force not sought, and equating the moments of the two forces considered.

55. Forces Acting at Different Points. Parallel Forces.—We have thus far considered forces acting upon a single particle, or upon one point of a body. If, however, two forces P and P', in the same plane, act upon A and B, two different points of a rigid body, they may still have a resultant.

FIG. 24.



Let the lines of directions of the two forces AF and BD (Fig. 24) be produced to meet in C. The two forces may then be considered as acting at C, and thus compounded into a single force at that point, or at the point G of the body.

Calling the angle $BCG = \beta$ and $ACG = \alpha$ we have, projecting P' and P upon the line of R,

$$R = P' \cos\beta + P \cos\alpha \ldots (1).$$

When the forces become parallel, as $A\ F$ and $B\ E$, $\beta = 0$, and $\alpha = 0$, and (1) becomes

$$R = P' + P \ldots (2).$$

If the parallel forces act in opposite directions, as $A\ F$ and $B\ E'$, then $\alpha = 180°$, and $\beta = 0$, and (1) becomes

$$R = P' - P \ldots (3).$$ Hence,

The resultant of two parallel forces is in a direction parallel to them and equal to their algebraic sum.

56. Point of Application of the Resultant.—Let P and P' (Figs. 25, 26) be two parallel forces acting in the same or in

Fig. 25.

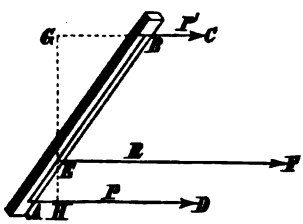
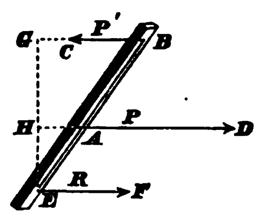


Fig. 26.



opposite directions, and let E be the point of application of the resultant. Assume this point as a centre of moments; then from Art. 54, since $L = 0$,

$P \times H\ E = P' \times G\ E$, or, in the form of a proportion,

$P' : P :: H\ E : G\ E$. But by similar triangles,

$H\ E : G\ E :: A\ E : E\ B$; $\therefore$

$P' : P :: A\ E : E\ B$.

That is, *the line of direction of the resultant of two parallel forces divides the line joining the points of application of the components, inversely as the components.*

By composition (Fig. 25) and division (Fig. 26) we obtain

$$P' + P : P :: A\ B : E\ B, \text{ and}$$

$$P - P' : P :: A\ B : E\ B.$$

That is, *if a straight line be drawn to meet the lines of two parallel forces and their resultant, each of the three forces will be proportional to that part of the line contained between the other two.*

When the forces act in the same direction, we have

$E\ B = \dfrac{P \times A\ B}{P' + P}$, and when they act in opposite directions,

$$E\ B = \frac{P \times A\ B}{P - P'}.$$

If, in the last case, $P = P'$, then $E\ B$ will be infinite. The two forces in this case constitute what is called a *couple.* Their effect is to produce rotation about a point between them.

Any number of parallel forces may be reduced to a single force

(or to a couple) by first finding the resultant of two forces, then the resultant of that and a third force, and so on to the last. And any single force may be resolved into two or any number of parallel forces by a method the reverse of this.

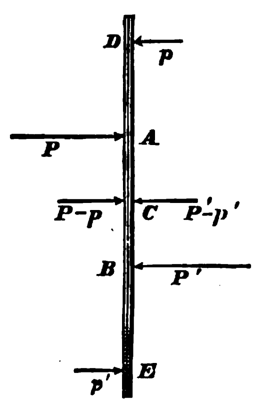
57. Equilibrium of Parallel Forces.—In order that a force may be in equilibrium with two parallel forces,

1. *It must be parallel to them.*
2. *It must be equal to their algebraic sum.*
3. *The distances of its line of action from the lines in which the two forces act, must be inversely as the forces.*

These three conditions belong to the *resultant* of two parallel forces, and therefore belong to that force which is in equilibrium with the resultant.

58. Equilibrium of Couples.—If two parallel forces are such as to constitute a couple, no *one* force can be in equilibrium with them. For the resultant of a couple is zero, and has its point of application at an infinite distance (Art. 56). But a couple can be held in equilibrium by another couple; and the second couple may be either larger or smaller than the given couple, or it may be equal to it.

FIG. 27.



Let the couple P and P' (Fig. 27) act on a body at the points A and B; they tend to produce rotation about the middle point C. If another couple, Q and Q', equal to P and P', should be applied to produce equilibrium, one must directly oppose P, and the other P'. Then A and B, being each held at rest, all the forces are in equilibrium.

But if the second couple is less than P and P', they must act at distances from C, which are as much greater as the forces are less; or, if the second couple is greater than the first, they must act at distances which are as much less. Thus, the couple p and p', acting at D and E, tend to produce rotation about C in one direction, and P and P' in the opposite; and these tendencies are equal when $D\,C : A\,C :: P : p$. For, since the opposite forces, P and p, are inversely as their distances from C, their resultant is at C, and is equal to $P - p$ (Art. 55). For the same reason, the resultant of P' and p' is at C, and equal to $P' - p'$. But $P - p = P' - p'$, and they act in opposite directions. Hence C is at rest, and therefore all the forces are in equilibrium.

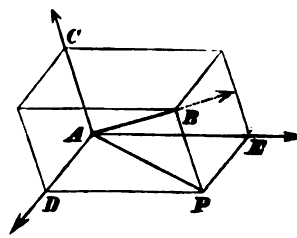
59. The Parallelopiped of Forces.—Hitherto forces have

been considered as acting in the same plane. But if forces act in different planes, the solution of every case may be reduced to the following principle, called the *parallelopiped of forces.*

Any three forces acting in different planes upon a body may be represented by the adjacent edges of a parallelopiped, and their resultant by the diagonal which passes through the intersection of those edges.

Let $A\ C$, $A\ D$, and $A\ E$ (Fig. 28), be three forces applied in different planes to the body at A. Construct the parallelopiped $C\ P$, having $A\ C$, $A\ D$, and $A\ E$, for its adjacent edges, and from A draw the diagonal $A\ B$. The section through the opposite edges $A\ C$ and $P\ B$ is a parallelogram, and therefore $A\ B$ is the resultant of $A\ C$ and $A\ P$, and $A\ P$ is the resultant of $A\ D$ and $A\ E$. Hence $A\ B$ is the resultant of $A\ C$, $A\ D$, and $A\ E$.

FIG. 28.



This process may obviously be reversed, and a given force may be resolved into three components in different planes along the edges of a parallelopiped, having such inclinations as we please.

60. Rectangular Axes.—The parallelopiped generally chosen is that whose sides are rectangles; the three adjacent edges of such a solid are called *rectangular axes.* All the forces which can possibly act on a body may be resolved into equivalent forces in the direction of three such axes. And since all forces which act in the direction of any one line may be reduced to a single force by taking their algebraic sum, therefore any number of forces acting through one point may be reduced to *three* in the direction of three axes chosen at pleasure.

FIG. 29.

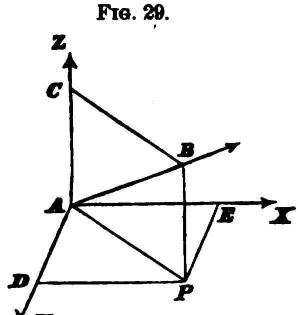
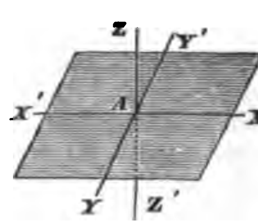


FIG. 30.



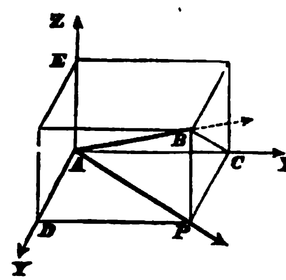
Let $A\ X$, $A\ Y$ (Fig. 29) be at right angles with each other, and $A\ Z$ perpendicular to the plane $A\ X$ and $A\ Y$. Let $A\ B$

represent a force acting on A: Resolve $A B$ into $A C$ on the axis $A Z$, and $A P$ in the plane of $A X$, $A Y$; then resolve $A P$ into $A D$ and $A E$ on the other two axes. Therefore, $A C$, $A D$, and $A E$ are three rectangular forces, whose resultant is $A B$.

Let the axes $A X$, $A Y$, $A Z$, be produced indefinitely (Fig. 30) to X', Y', Z'; then their planes will divide the angular space about A into eight solid right angles, namely: A-$X Y Z$, A-$X Y' Z$, A-$X' Y' Z$, A-$X' Y Z$, above the plane of X and Y, and A-$X Y Z'$, A-$X Y' Z'$, A-$X' Y' Z'$, A-$X' Y Z'$ below it.

61. Geometrical Relation of Components and Resultant.—A force acting on the body A may be situated in any one of the eight angles, and its value may be expressed in terms of the squares of its three components. Let $A B$ (Fig. 31) be resolved as before into the rectangular components $A C$, $A D$, and $A E$. Then, by the right-angled triangles, we find

FIG. 31.



$$A B^2 = B P^2 + A P^2 = A E^2 + A P^2;$$
and
$$A P^2 = A C^2 + C P^2 = A C^2 + A D^2;$$

$$\therefore A B^2 = A C^2 + A D^2 + A E^2;$$
$$\text{and} \quad A B = \sqrt{A C^2 + A D^2 + A E^2}.$$

If $A B$ is in the plane of X and Y, the component on the axis of Z becomes zero, and $A B = \sqrt{A C^2 + A D^2}$, and similarly for the other planes.

62. Trigonometrical Relation of Components and Resultant.—Let the angles which $A B$ makes with the axes of X, Y, Z, respectively, be α, β, γ; that is, $B A C = \alpha$, $B A D = \beta$, $B A E = \gamma$. In the triangle $A B C$, right-angled at C, we have

$$A C = A B \, . \cos \alpha.$$

In like manner,

$$A D = A B \, . \cos \beta;$$
$$\text{and} \quad A E = A B \, . \cos \gamma.$$

And since $A B$ is the resultant of the forces $A C$, $A D$, and $A E$, it is the resultant of $A B \, . \cos \alpha$, $A B \, . \cos \beta$, $A B \, . \cos \gamma$. In general, the components of any force P, when resolved upon three rectangular axes, are $P \, . \cos \alpha$, $P \, . \cos \beta$, $P \, . \cos \gamma$.

63. Any Number of Forces Reduced to Three on Three Rectangular Axes.—Suppose the body at A to be acted upon by a second force P', whose direction makes with the axes the angles α', β', γ'; then, as before, P' is the resultant of $P' . \cos\alpha'$, $P' . \cos\beta'$, $P' . \cos\gamma'$; and a third force P'', in like manner, has for its components $P'' . \cos\alpha''$, $P'' . \cos\beta''$, $P'' . \cos\gamma''$; and so of any number of forces.

Now, all the components on one axis may be reduced to one force by adding them together. Hence, the whole force in the axis of $X = P . \cos\alpha + P' . \cos\alpha' + P'' . \cos\alpha'' + P''' . \cos\alpha''' + \&c.$; the whole in the axis of Y,

$$= P . \cos\beta + P' . \cos\beta' + P'' . \cos\beta'' + P''' . \cos\beta''' + \&c.;$$

and that in the axis of Z,

$$= P . \cos\gamma + P' . \cos\gamma' + P'' . \cos\gamma'' + P''' . \cos\gamma''' + \&c.$$

If any component acts in a direction opposite to others in the same axis, it is affected by a contrary sign, so that the force in the direction of any axis is the algebraic sum of all the individual forces in that axis.

If the sum of the components in *one* axis is reduced to zero by contrary signs, the effect of all the forces is limited to the plane of the other axes, and is to be obtained as in Art. 52, where two axes were employed. If the sum of the components on each of *two* axes is reduced to zero, then the whole force is exerted in the direction of the remaining axis, and is therefore perpendicular to the plane of the other two.

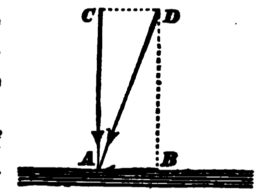
64. Equilibrium of Forces in Different Planes.—Since all the forces which can operate on a body may be reduced to three forces on rectangular axes, it is obvious that the whole system of forces cannot be in equilibrium till the sum of the components on each axis is reduced to zero. We must have, therefore, in Art. 63, as conditions of equilibrium, these three equations for the three axes, X, Y, Z:

$$P . \cos\alpha + P' . \cos\alpha' + P'' . \cos\alpha'' +, \&c., = 0;$$
$$P . \cos\beta + P' . \cos\beta' + P'' . \cos\beta'' +, \&c., = 0;$$
$$P . \cos\gamma + P' . \cos\gamma' + P'' . \cos\gamma'' +, \&c., = 0.$$

65. Forces Resisted by a Smooth Surface.—Whenever any forces cause pressure upon a surface without friction, and are held in equilibrium by its resistance, the resultant of those forces must be at right angles to the surface. Suppose that

$D A$ (Fig. 32) is either a single force or the resultant of two or more forces, and that it is held in equilibrium by the reaction of $A B$, a smooth surface. If $D A$ is not perpendicular to the surface, it can be resolved into two components, one perpendicular to the surface $A B$, the other parallel to it. The former, $C A$, is neutralized by the resistance of the surface; the latter, $B A$, is not resisted, and produces motion parallel to the surface, contrary to the supposition. Therefore $D A$, if held in equilibrium by the surface $A B$, must be perpendicular to it.

FIG. 32.



CHAPTER IV.

THE CENTRE OF GRAVITY.

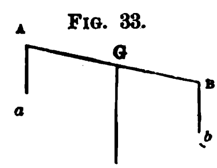
66. The Centre of Gravity Defined.—Whenever gravity pulls a body towards the earth, the pull is the resultant of the parallel forces exerted by the earth upon the separate particles of the body. Whatever the position of the body this resultant passes through a certain point called the *centre of gravity*. Hence,

The centre of gravity of a body is that point at which the whole mass of the body may be considered as concentrated; or,

It is the point at which the body, if supported there, and if acted upon by gravity alone, will balance in every position.

67. Centre of Gravity of Equal Bodies in a Straight Line.—The centre of gravity of two equal particles is in the middle point between them. Let A and B (Fig. 33), two equal particles, be joined by a straight line, and let $A a$ and $B b$ represent the forces of gravity. The resultant of these forces, since they are parallel and equal, will pass through the middle of $A B$ (Art. 56); G is therefore the centre of gravity. In like manner it is proved that the centre of gravity of two equal *bodies* is in the middle point between their respective centres of gravity.

FIG. 33.



Any number of equal particles or bodies, arranged at equal distances on a straight line, have their common centre of gravity in the middle; since the above reasoning applies to each pair, taken at equal distances from the extremes. Hence, the centre of gravity of a material straight line (e.g., a fine straight wire) is in the middle point of its length.

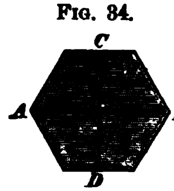
68. Centre of Gravity of Regular Figures.—In the discussion of the centre of gravity in relation to *form*, *bodies* are considered uniformly dense, and *surfaces* are regarded as thin laminæ of matter.

In plane figures the centre of gravity coincides with the centre of magnitude, when they have such a degree of regularity that there are two diameters, each of which divides the figure into equal and symmetrical parts.

The circle, the parallelogram, the regular polygon, and the ellipse, are examples.

For instance, the regular hexagon (Fig. 34) is divided symmetrically by $A\,B$, and also by $C\,D$. Conceive the figure to be composed of material lines parallel to $A\,B$. Each of these has its centre of gravity in its middle point, that is, in $C\,D$, which bisects them all (Art. 67). Hence, the centre of gravity of the whole figure is in $C\,D$. For the same reason it is in $A\,B$. It is, therefore, at their intersection, which is also the centre of magnitude.

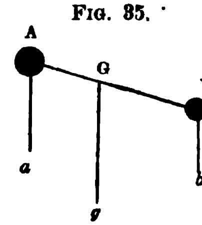
Fig. 34.



By a similar course of reasoning it is shown that in *solids* of uniform density, which are so far regular that they can be divided symmetrically by three different planes, the centres of gravity and magnitude coincide; e.g., the sphere, the parallelopiped, the cylinder, the regular prism, and the regular polyhedron.

69. Centre of Gravity between Two Unequal Bodies.—The centre of gravity of two unequal bodies is in a straight line joining their respective centres of gravity, and at the point which divides their distance in the inverse ratio of their weights. Let $A\,a$ and $B\,b$ (Fig. 35), passing through the centres of gravity of A and B, be proportional to their weights, and therefore represent the forces of gravity exerted upon them. By the laws of parallel forces, the resultant $G\,g = A\,a + B\,b$ (Art. 55), and $A\,a : B\,b :: B\,G : A\,G$. Therefore the centre of gravity must be at G, through which the resultant passes

Fig. 35.



(Art. 66). This obviously includes the case of *equal* weights (Art. 67).

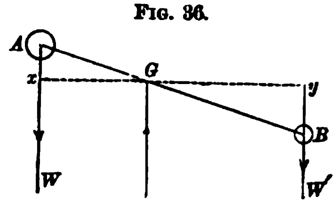
It appears from the foregoing that the whole pressure on a support at G is $A + B$, and that the system is kept in equilibrium by such support.

70. Equal Moments with Respect to the Centre of Gravity.—Applying the principle of moments we have, calling the weights W and W', and taking the centre of moments at G (Fig. 36)

$$W \times Gx = W' \times Gy;$$

but $Gx : Gy :: AG : GB$;
$\therefore W \times AG = W' \times GB$, as was proved in Art. 56.

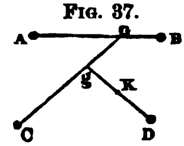
FIG. 36.



71. Centre of Gravity between Three or More Bodies.—The method of determining the centre of gravity of two bodies may be extended to any number.

Let A, B, C, D, &c. (Fig 37), be the weights of the bodies, and let the centres of gravity of A and B be connected together by the inflexible line AB.

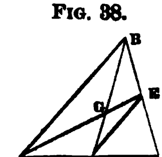
FIG. 37.



Divide AB so that $A : B :: BG : AG$, or $A + B : B :: AB : AG$; then G is the centre of gravity of A and B. Join CG; and since $A + B$ may be considered as at the point G, divide CG so that $A + B + C : C :: CG : Gg$. In like manner, K, the centre of gravity of four bodies, is found by the proportion, $A + B + C + D : D :: Dg : gK$. The same plan may be pursued for any number of bodies.

72. Centre of Gravity of a Triangle.—*The centre of gravity of a triangle is one-third of the distance from the middle of a side on a line to the opposite angle.* Bisect AC in D (Fig. 38), and BC in E. BD bisects all lines across the triangle parallel to AC; therefore the centre of gravity of all those lines—that is, of the triangle—is in BD. For a like reason, it is in AE, and therefore at their intersection, G. Since $EC = \frac{1}{2} BC$, and $DC = \frac{1}{2} AC$, $\therefore ED = \frac{1}{2} AB$. But EGD and AGB are similar; $\therefore DG : BG :: DE : AB :: 1 : 2$; $\therefore DG = \frac{1}{2} BG = \frac{1}{3} BD$.

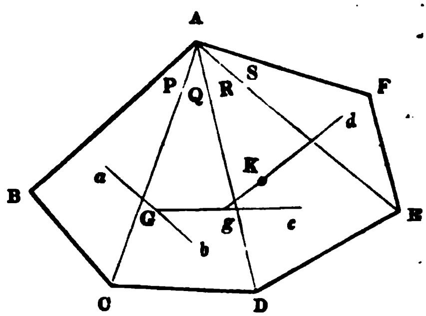
FIG. 38.



73. Centre of Gravity of an Irregular Polygon.—Divide the polygon into triangles by diagonals drawn through one of its

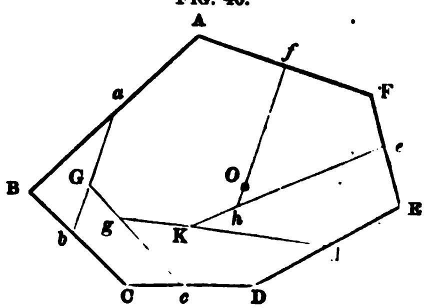
angles, and then proceed according to the methods already given. Let $A\,C\,E$ (Fig. 39) be an irregular polygon, whose centre of gravity is to be found. Divide it into the triangles P, Q, R, S, by diagonals through A, and find their centres of gravity a, b, c, d (Art. 72). Join $a\,b$, and divide it so that $a\,b : a\,G :: P + Q : Q$; then G is the centre of gravity of the quadrilateral $P + Q$. Then join $G\,c$, and make $G\,c : G\,g :: P + Q + R : R$. By proceeding in this manner till all the triangles are used, the centre of gravity of the polygon is found at the last point of division.

FIG. 39.



74. Centre of Gravity of the Perimeter of an Irregular Polygon.—Find the centre of gravity of each side, which is at its middle point, and then proceed as in Art. 71, the weight of each line being considered proportional to its length. Thus, let a, b, c, &c., be the centres of gravity of the sides, $A\,B$, $B\,C$, $C\,D$, &c. (Fig. 40); join $a\,b$, and divide it so that $a\,b : a\,G :: A\,B + B\,C : B\,C$; then G is the centre of gravity of $A\,B$ and $B\,C$. Next join $G\,c$, and make $G\,c : G\,g :: A\,B + B\,C + C\,D : C\,D$; then g is the centre of gravity of those three sides. Proceed in this manner till all the sides are used.

FIG. 40.

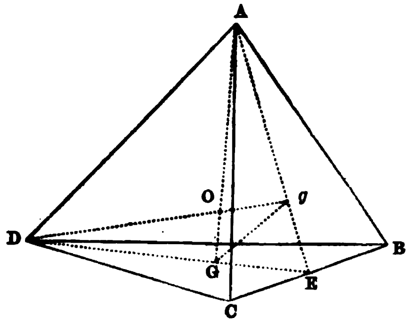


The perimeter of a polygon having the degree of regularity described in Art. 68, has its centre of gravity at the centre of the figure, as may be easily proved. If a polygon has a less degree of regularity than that, the centre of gravity both of its area and its perimeter may usually be found by methods more direct and simple than those given for polygons wholly irregular.

75. Centre of Gravity of a Pyramid.—*The centre of gravity of a triangular pyramid is in the line joining the vertex and the centre of gravity of the base, at one-fourth of the distance from the base to the vertex.*

Let G (Fig. 41) be the centre of gravity of the base $B\,D\,C$; and g that of the face $A\,B\,C$. The line $A\,G$ passes through the centre of gravity of every lamina parallel to $D\,B\,C$, on account of the similarity and similar position of all those laminæ; $\therefore$ the centre of gravity of the pyramid is in $A\,G$. For a similar reason, it is in $D\,g$; and therefore at their intersection, O. Now $EG = \frac{1}{3}\,E\,D$, and $E\,g = \frac{1}{3}\,E\,A$; hence, by similar triangles, $g\,G = \frac{1}{3}\,A\,D$. But $G\,g\,O$ and $A\,O\,D$ are also similar; $\therefore$ $G\,O = \frac{1}{3}\,A\,O = \frac{1}{4}\,A\,G$.

Fig. 41.

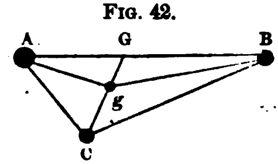


From this it is readily proved that the centre of gravity of *every* pyramid and cone is one-fourth of the distance from the centre of gravity of the base to the vertex.

76. Examples on the Centre of Gravity.—

1. A, B, and C (Fig. 42), weigh, respectively, 3, 2, and 1 pounds, $A\,B = 5$ ft., $B\,C = 4$ ft., and $C\,A = 2$ ft. Find the distance of their centre of gravity from C.

Fig. 42.



First, from the given sides of the triangle $A\,B\,C$, calculate the angles. A is found to be $49°\ 27\frac{1}{2}'$. Next find the place of G, the centre of gravity of A and B, by the proportion, $A + B : B :: A\,B : A\,G$; $A\,G$ is 2 ft., equal to $A\,C$. Calculate $C\,G$, the base of the isosceles triangle $A\,G\,C$. Its length is 1.673. Then find $C\,g$ by the proportion $C\,G : C\,g :: A + B + C : A + B$; therefore $C\,g = 1.394$.

2. $A = 5$ lbs., $B = 3$ lbs., and $C = 12$ lbs.; $A\,B = 8$ ft., $A\,C = 4$ ft., and the angle A is $90°$; find the distance of the centre of gravity of A, B, and C from C. *Ans.* 2 ft.

3. Three equal bodies are placed at the angles of any triangle whatever; show that the common centre of gravity of those bodies coincides with the centre of gravity of the triangle.

4. Find the centre of gravity of five equal heavy particles placed at five of the angular points of a regular hexagon.

Ans. It is one-fifth of the distance from the centre to the third particle.

5. A regular hexagon is bisected by a line joining two opposite angles; where is the centre of gravity of one-half?

Ans. Four-ninths of the distance from the centre to the middle of the second side.

6. A square is divided by its diagonals into four equal parts, one of which is removed; find the distance from the opposite side of the square to the centre of gravity of the remaining figure.

Ans. $\frac{7}{18}$ of the side of the square.

7. Two isosceles triangles are constructed on opposite sides of the same base, the altitude of the greater being h, and of the less, h'; where is the centre of gravity of the whole figure?

Ans. On the altitude of the greater triangle, at a distance from the common base equal to $\frac{1}{3}(h - h')$.

8. The base and the place of the centre of gravity of a triangle being given, required to construct the triangle.

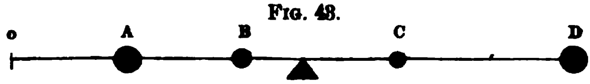
9. Given the base and altitude of a triangle; required to construct the triangle, when its centre of gravity is perpendicularly over one end of the base.

10. On a cubical block stands a square pyramid, whose base, volume, and mass are respectively equal to those of the cube; where is the centre of gravity of the figure?

Ans. One-eighth of the height of the cube above its upper surface.

77. Centre of Gravity of Bodies in a Straight Line referred to a Point in that Line.—If several bodies are in a straight line, their common centre of gravity may be referred to a point in that line; and its distance from that point is obtained by *multiplying each weight into its own distance from the same point, and dividing the sum of the products by the sum of the weights.* Let A, B, C, and D, represent the weights of several bodies, whose centres of gravity are in the straight line $o\ D$ (Fig. 43). Required

FIG. 43.



the distance of their common centre of gravity from any point o assumed in the same line. Let G be their common centre of gravity; then, calling R the resultant of the several weights A, B, C and D, which acts at the point G, we have from principle of moments,

$$R \times o\,G = A \times A\,o + B \times B\,o + C \times C\,o + D \times D\,o,$$

and since $R = A + B + C + D$ (Art 55), we have

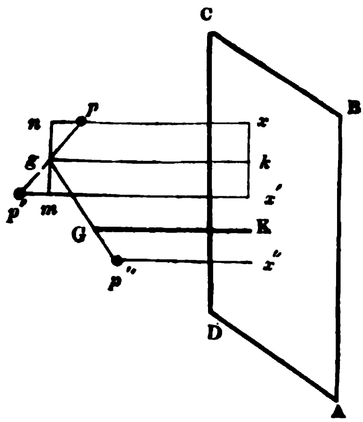
$$o\,G = \frac{A \times A\,o + B \times B\,o + C \times C\,o + D \times D\,o}{A + B + C + D}.$$

78. Centre of Gravity of a System referred to a Plane.—If the bodies are not in a straight line, they may be referred to a plane, which is assumed at pleasure. The distance of

their common centre of gravity from that plane is expressed as before: *multiply each weight into its own distance from the plane, and divide the sum of the products by the sum of the bodies.*

Let p, p' p'' (Fig. 44), represent the weights of several bodies, whose centres of gravity are at those points respectively, and let $A\ C$ be the plane of reference.

FIG. 44.



Join $p\ p'$, and let g be the common centre of gravity of p and p'; draw $p\ x$, $g\ k$, $p'\ x'$ at right angles to the plane $A\ C$, and consequently parallel to each other; join $x\ x'$, and since the points p, g, p', are in a straight line, the points x, k, x' will also be in a straight line, and therefore $x\ x'$ will pass through k. Join $g\ p''$, and let G be the common centre of gravity of p, p', p''; draw $G\ K$, $p''\ x''$, perpendicular to the plane; and through g draw $m\ n$ parallel to $x\ x'$ meeting $p\ x$ produced in n.

Now $p : p' :: p'\ g : p\ g ::$ (by sim. triangles) $p'\ m : p\ n$;

$\therefore p \times p\ n = p' \times p'\ m$, or $p \times (n\ x - p\ x) = p' \times (p'\ x' - m\ x')$; but

$$n\ x = g\ k = m\ x', \ \therefore p \times (g\ k - p\ x) = p' \times (p'\ x' - g\ k),$$

and

$$(p + p') \times g\ k = p \times p\ x + p' \times p'\ x' \ \therefore g\ k = \frac{p \times p\ x + p' \times p'\ x'}{p + p'};$$

for the same reason, if $p + p'$ is placed at g, we have

$$G\ K = \frac{(p+p') \times g\ k + p'' \times p''\ x''}{(p+p') + p''} = \frac{p \times p\ x + p' \times p'x' + p'' \times p''x''}{p + p' + p''};$$

a formula which is applicable to any number of bodies.

Let the last equation be multiplied by the denominator of the fraction, and we have

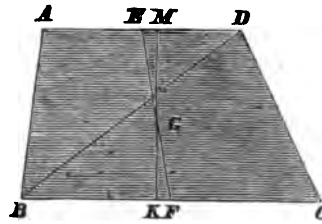
$$(p + p' + p'' + \&c.)\ G\ K = p \times p\ x + p' \times p'\ x' + p'' \times p''\ x'' + \&c.:$$

that is, *the moment of any system of bodies with reference to a given plane, equals the sum of the moments of all the parts of the system with reference to the same plane.*

79. Centre of Gravity of a Trapezoid.—As an example of the foregoing principle, let it be proposed to find the centre of

gravity of a trapezoid, considered as composed of two triangles. The centre of gravity of the trapezoid $A\ C$ (Fig. 45) is in $E\ F$, which bisects all the lines of the figure parallel to $B\ C$. Suppose G to be the centre of gravity of the trapezoid; through G draw $K\ M$ perpendicular to the bases. Let $K\ M = h$, $B\ C = B$, $A\ D = b$, and join $B\ D$.

FIG. 45.



The moment of the trapezoid with reference to $B\ C$ is

$$(B + b)\,\frac{h}{2} \cdot G\,K.$$

The moment of the upper triangle is $\frac{b\,h}{2} \cdot \frac{2}{3}h$; the moment of the lower triangle is $\frac{B\,h}{2} \cdot \frac{h}{3}$;

$$\therefore (B + b)\,\frac{h}{2} \cdot G\,K = \frac{B\,h}{2} \cdot \frac{h}{3} + \frac{b\,h}{2} \cdot \frac{2}{3}h; \text{ whence}$$

$$G\,K = \frac{B + 2\,b}{B + b} \cdot \frac{h}{3}. \quad \text{But } G\,M = h - \frac{B + 2\,b}{B + b} \cdot \frac{h}{3} =$$

$$\frac{2\,B + b}{B + b} \cdot \frac{h}{3}; \quad \therefore G\,M : G\,K :: 2\,B + b : B + 2\,b.$$

By similar triangles

$$G\,M : G\,K :: E\,G : G\,F; \therefore E\,G : G\,F :: 2\,B + b : B + 2\,b; \text{ or}$$

the centre of gravity of a trapezoid is on the line which bisects the parallel bases, and divides it in the ratio of twice the longer plus the shorter to twice the shorter plus the longer.

1. Four bodies, A, B, C, D, weighing, respectively, 2, 3, 6, and 8 pounds are placed with their centres of gravity in a right line, at the distance of 3, 5, 7, and 9 feet from a given point; what is the distance of their common centre of gravity from that given point; and between which two of the bodies does it lie?

Ans. Between C and D; and its distance from the given point $7\frac{4}{19}$ feet.

2. There are five bodies, weighing, respectively, 1, 14, $21\frac{1}{2}$, 22, and $29\frac{1}{2}$ pounds; a plane is assumed passing through the last body, and the distances of the other four from the plane are, respectively, 21, 5, 6, and 10 feet; how far from the plane is the common centre of gravity of the five bodies? *Ans.* 5 feet.

80. Centrobaric Mensuration.—The properties of the centre of gravity furnish a very simple method of measuring

surfaces and solids of revolution. This method is comprehended in the two following propositions, known as the theorems of Guldinus:

1. *If any line revolve about a fixed axis, which is in the plane of that line, the* SURFACE *which it generates is equal to the product of the given line into the circumference described by its centre of gravity.*

Let any line, either straight or curved, revolve about a fixed axis which is in the plane of that line; and let f, f', f'', f''', etc., denote elementary portions of the line, d, d', d'', d''', &c., the distances of these portions, respectively, from the axis; then the surface generated by f, in one revolution, will be $2\pi d f$; hence the surface generated by the whole line will be

$$S = 2\pi\,(d f + d' f' + d'' f'' + d''' f''' + \&c.) \ldots (1).$$

Put L = the length of the revolving line, and G = the distance from the axis to the centre of gravity of the line; then (Art. 78)

$$G\,L = d f + d' f' + d'' f'' + d''' f''' + \&c.. \ldots (2).$$

Combining (1) and (2), we have

$$S = 2\pi\,G\,L \ldots\ldots\ldots (3).$$

2. *If a plane surface, of any form whatever, revolve about a fixed axis which is in its own plane, the* VOLUME *generated is equal to the product of that surface into the circumference described by its centre of gravity.*

Let any plane surface revolve about an axis which is in the plane of that surface; and let f, f', f'', f''', &c., denote elementary portions of the surface, d, d', d'', d''', &c., the distances of these portions, respectively, from the axis; then the volume generated by f in one revolution will be $2\pi d f$; hence the volume generated by the whole surface will be

$$V = 2\pi\,(d f + d' f' + d'' f'' + d''' f''' + \&c.) \ldots (4).$$

Put A = the area of the revolving surface, and G = the distance from the axis to the centre of gravity of that surface; then (Art. 78)

$$A\,G = d f + d' f' + d'' f'' + d''' f''' + \&c., \ldots (5).$$

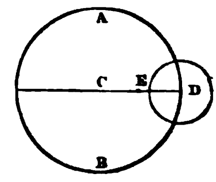
Substituting in (4), we have

$$V = 2\pi\,A\,G \ldots\ldots\ldots (6).$$

As an illustration of the first theorem, the straight line $C\,D$ (Fig. 46), revolving about the center C, describes a circle whose

surface is equal to $C D$ into the circumference of the circle described by its centre of gravity, E. This is evident also from the consideration that, since E is the centre of the line $C D$, the circumference described by it will be half the length of the circumference $A D B$; and the area of a circle is equal to the product of the radius into half the circumference.

Fig. 46.



The second theorem is illustrated by the volume of a cylinder, whose height $= h$, and the radius of whose base $= r$.

Common method; base$=\pi r^2$; height$=h$; $\therefore$ vol. $= \pi r^2 h$.

Centrobaric method; revolving area $= r h$; circumference described by the centre of gravity $= \frac{1}{2} r \times 2 \pi$; $\therefore$ vol. $= r h \,.\, \frac{1}{2} r \,.\, 2 \pi = \pi r^2 h$.

81. Examples.—

1. Suppose the small circle (Fig. 46) to be placed with its plane perpendicular to the plane of the paper, and revolved about C, the point D describing the line $D B A$; required the content of the solid ring. If $C D = R$, and $E D = r$, then the area revolved $= \pi r^2$, and the circumference $D B A = 2 \pi R$; $\therefore$ the ring $= 2 \pi^2 R r^2$. It is equal to a cylinder whose base is the circle $E D$, and whose height equals the line $D B A$.

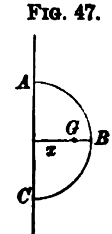
2. Find the convex surface of a cone; slant height $= s$; and rad. of base $= r$. The line revolved being s, and the distance from the axis to its centre of gravity, $\frac{1}{2} r$, the surface is $\pi r s$.

3. A square, whose side is one foot, is revolved about an axis which passes through one of its angles, and is parallel to a diagonal; required the volume of the figure thus formed.

Ans. $\pi \sqrt{2}$, or 4.4429 cubic ft.

4. Find the centre of gravity of a semi-circumference. In this case the revolving semi-circumference $A B C$ (Fig. 47) generates the surface of a sphere; hence, taking the diameter $A C$ as an axis, calling the distance of the centre of gravity G from the axis x and radius r, we have

Fig. 47.



$$4 \pi r^2 = 2 \pi x \times \pi r; \;\therefore x = \frac{2 r}{\pi}.$$

Hence the distance of the centre of gravity of the semicircumference from the centre of the circle is

$$\frac{2 r}{\pi} = .637 \, r.$$

5. Find the centre of gravity of a semicircle. The revolving

area generates a sphere, and hence, as in the preceding problem, we have

$$\tfrac{4}{3}\,\pi\,r^3 = 2\,\pi\,x \times \tfrac{1}{2}\,\pi\,r^2\,;\ \therefore\ x = \frac{4}{3}\,\frac{r}{\pi} = .424\,r.$$

In any case, when a simple expression for the surface generated by a revolving line can be found, it is easy to find the centre of gravity of the line by this method, and the centre of gravity of an area may be readily found from the expression of the volume generated.

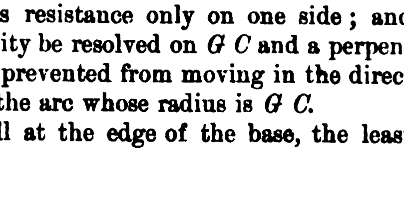
82. Support of a Body.—A body cannot rest on a smooth plane unless it is horizontal; for the pressure on a plane (Art. 65) cannot be balanced by the resistance of that plane, except when perpendicular to it; therefore, as the force of gravity is vertical, the resisting plane must be horizontal.

The *base of support* is that area on the horizontal plane which is comprehended by lines joining the extreme points of contact.

If there are *three* points of contact, the base is a triangle; if *four*, a quadrilateral, &c.

When the vertical through the centre of gravity (called *the line of direction*) falls within the base, the body is supported; if without, it is not supported. In the body A (Fig. 48) the force of gravity acts in the line $G\ F$, and there are lines of resistance on both sides of $G\ F$, as $G\ C$ and $G\ E$, so that the body cannot turn on the edge of the base, without *rising* in an arc whose radius is $G\ C$ or $G\ E$.

FIG. 48.



But, in the body B, there is resistance only on one side; and therefore, if the force of gravity be resolved on $G\ C$ and a perpendicular to it, the body is not prevented from moving in the direction of the latter, that is, in the arc whose radius is $G\ C$.

If the line of direction fall at the edge of the base, the least force will overturn it.

83. Different Kinds of Equilibrium.—If the base is reduced to a line or point, then, though there may be support, there is no *firmness* of support; the body will be moved by the least force. But it is affected very differently in different cases.

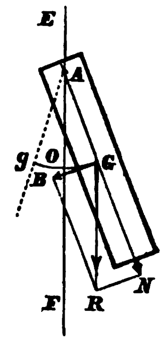
When it is moved from its position of support and left, it will in some cases return to it, pass by, and return again, and continue thus to vibrate till it settles in its place of support by friction and other resistances. This condition is called *stable equilibrium*.

In other cases, when moved from its position of support and left, it will depart further from it, and never recover that position again. This is called *unstable equilibrium.*

In other cases still, the body, when moved from its place of support and left, will remain, neither returning to it nor departing further from it. This is called *neutral equilibrium.*

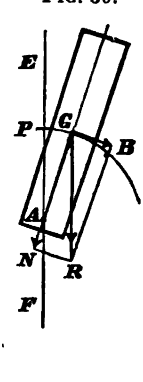
84. Stable Equilibrium.—Let the body (Fig. 49) be suspended on the pivot A. This is its base of support. While the centre of gravity is below A, the line of direction $E\ O\ F$ passes through the base, and the body is supported. Let it be moved aside, and the centre of gravity be left at G. Let $G\ R$ represent the force of gravity, and resolve it into $G\ N$ on the line $A\ G$, and $N\ R$, or $G\ B$, perpendicular to $A\ G$. $G\ N$ is resisted by the strength of A, and $G\ B$ moves the centre of gravity in the arc whose radius is $A\ G$. Hence the body swings with accelerated motion till the centre of gravity reaches O, where the force $G\ B$ becomes zero. But by its inertia, the body passes beyond that position, and ascends on the other side, till the retarding force of gravity stops it at g, as far from O as G is. It then descends again, and would never cease to oscillate were there no obstructions.

FIG. 49.



85. Unstable Equilibrium.—Next, let the body be turned on the pivot till the centre of gravity G is at P, above A (Fig. 50). Then, as well as when G is below A, the body is supported, because the line of direction $E\ P\ F$ passes through the base A. But if turned and left in the slightest degree out of that position, it cannot recover it again, but will depart further and further from it. Let $G\ R$ represent the force of gravity, and let it be resolved into $G\ N$, acting through A, and $G\ B$ perpendicular to it. The former is resisted by A; the latter moves G away from P, the place of support. If the body is free to revolve about A, without falling from it, the centre of gravity will, by friction and other resistances, finally settle below A, as in the case of stable equilibrium.

FIG. 50.

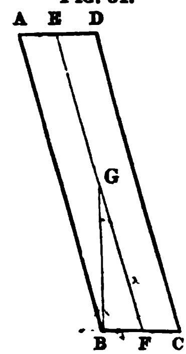


86. Neutral Equilibrium.—Once more, suppose the pivot supporting the body to be at G, the centre of gravity; then, in

whatever situation the body is left, the line of direction passes through the base, and the body rests indifferently in any position.

These three kinds of equilibrium may be illustrated also by bodies resting by curved surfaces on a horizontal plane. Thus, if a cylinder is uniformly dense, it will always have a *neutral* equilibrium, remaining wherever it is placed. But if, on account of unequal density, its centre of gravity is not in the axis, then its equilibrium is *stable*, when the centre of gravity is below the axis, and *unstable* when above it.

FIG. 51.



In general, there is stable equilibrium when the centre of gravity, on being disturbed in either direction, begins to *rise;* unstable when, if disturbed either way, it begins to *descend;* and neutral when the disturbance neither raises nor lowers the centre of gravity.

87. Questions on the Centre of Gravity.—

1. A frame 20 feet high, and 4 feet in diameter, is racked into an oblique form (Fig. 51), till it is on the point of falling; what is its inclination to the horizon? *Ans.* 78° 27′ 47″.

2. A stone tower, of the same dimensions as the former, is inclined till it is about to fall, but preserves its rectangular form; what is its inclination? *Ans.* 78° 41′ 24″.

3. A cube of uniform density lies on an inclined plane, and is prevented by friction from sliding down; to what inclination must the plane be tipped, that the cube may just begin to roll down? *Ans.* 45°.

4. What must be the inclination of a plane, in order that a regular prism of any given number of sides may be on the point of rolling down, if friction prevents sliding?

Ans. Equal to half the angle at the centre of the prism, subtended by one side.

5. A body weighing 83 lbs. is suspended, and drawn aside from the vertical 9° by a horizontal force; what pressure is there on the point of support, and what force urges it down the arc?

Ans. Pressure on the support, 84.03 lbs.
Moving force, 12.984 lbs.

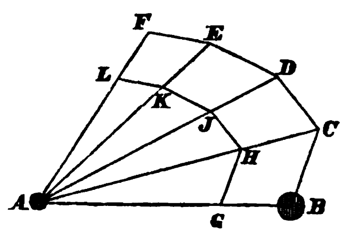
88. Motion of the Centre of Gravity of a System when one of the Bodies is Moved.—

When one body of a system is moved, the centre of gravity of the system moves in a similar path, and its velocity is to that of

the moving body as the mass of that body is to the mass of the whole system.

If the system contains but two bodies, A and B (Fig. 52), suppose A to remain at rest, while B describes the straight lines $B\ C$, $C\ D$, &c., the centre of gravity G will in the same time describe the similar series, $G\ H$, $H\ J$, &c. When B is in the position B, and the centre of gravity at G, $A\ G : A\ B :: B : A + B$; when B is at C, $A\ H : A\ C :: B : A + B$; $\therefore A\ G : A\ B :: A\ H : A\ C$. Hence $G\ H$ is parallel to $B\ C$, and $G\ H : B\ C :: B : A + B$. In like manner, $H\ J : C\ D :: B : A + B$, &c. Thus all the parts of one path are parallel to the corresponding parts of the other, and have a constant ratio to them. Therefore the paths are similar. As the corresponding parts are described in equal times, their lengths are as the velocities. But the lengths are as $B : A + B$; therefore the velocity of the common centre of gravity is to that of the moving body as the mass of the moving body is to the mass of both. The same reasoning is applicable when the body moves in a curve.

FIG. 52.



If the system contain any number of bodies, and the centre of gravity of the whole be at G, then the centre of gravity of all except B must be in the line $B\ G$ beyond G. Suppose it to be at A, and to remain at rest, while B moves; then it is proved in the same manner as before, that G, the centre of gravity of the whole system, moves in a path parallel to the path of B, and with a velocity which is to B's velocity as the mass of B to the mass of the entire system.

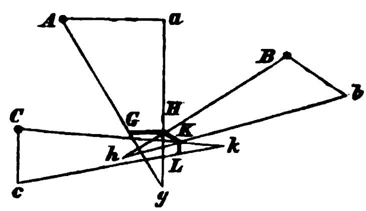
89. Motion of the Centre of Gravity of a System when Several of the Bodies are Moved.—

When any or all of the bodies of a system are moved, the centre of gravity moves in the same manner as if all the system were collected there, and acted on by the forces which act on the separate bodies.

Let A, B, C, &c. (Fig. 53), belong to a system containing any number of bodies, and let M be the mass of the system. Let A be moved over $A\ a$, B over $B\ b$, C over $C\ c$, &c. And first suppose the motions to be made in equal successive times. If the centre of gravity of the system is first at G, then that of all the bodies except A is in $A\ G$ produced, as at g. While A moves to a, G moves in a parallel line to H (Art. 88), and $G\ H : A\ a :: A : M$. In like manner, when B describes $B\ b$, the centre of gravity of the other bodies being at h, the centre of gravity of the system de-

scribes the parallel line, $H K$, and $H K : B b :: B : M$; and when C moves, $K L : C c :: C : M$, &c. Now, $A a$ and $G H$ represent the respective velocities of the body A, and the system M; therefore, if we convert the proportion $G H : A a :: A : M$ into an equation, we have $A \times A a = M \times G H$; that is, the momentum of the body A equals the momentum of the system M. It therefore requires the same force to move A over $A a$ as to move the system M over $G H$. The same is true of the other bodies. If then the several forces which move the bodies, limiting the number to three, for the present, were applied successively to the system collected at G, they would move it over $G H$, $H K$, $K L$. But if applied at once, they would move it over $G L$, the remaining side of the polygon. If, therefore, the forces, instead of acting successively on the bodies, were to move A over $A a$, B over $B b$, and C over $C c$, at the same time, the centre of gravity of the system would describe $G L$ in the same time. In the same way it may be proved, that whatever forces are applied to the several bodies of a system, the centre of gravity of the system is moved in the same manner as a body equal to the whole system would be moved, if all the same forces were applied to it.

Fig. 53.



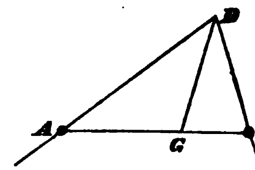
It is possible that the centre of gravity of a system should remain at rest, while all the bodies in it are in motion. For, suppose all the forces acting on the bodies to be such that they might be represented in direction and intensity by all the sides of a polygon, then, since a single body acted on by them would be in equilibrium, therefore the centre of gravity of the system would remain at rest, though the bodies composing it are in motion.

90. Mutual Action among the Bodies of a System.—The forces which have been supposed to act on the several bodies of a system are from without, and not forces which some of the bodies within the system exert on others. If the bodies of a system mutually attract or repel each other, such action cannot affect the centre of gravity of the whole system. For action and reaction are always opposite and equal. Whatever force one body exerts on any other to move it, that other exerts an equal force on the first, and the two actions produce equal and opposite effects on the centre of gravity between them. Therefore the centre of gravity of a system remains at rest, if the bodies which compose it are acted on only by their mutual attractions or repulsions.

91. Examples on the Motion of the Centre of Gravity.—

1. Two bodies, *A* and *B*, of given weights, start together from *D* (Fig. 54), and move uniformly with given velocities in the directions *D A* and *D B*; required the direction and velocity of their centre of gravity.

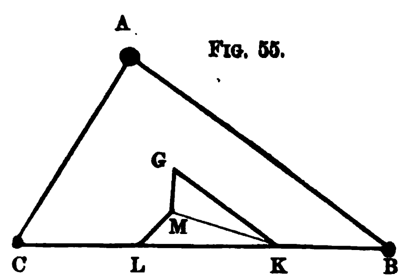
FIG. 54.



As the directions of *D A* and *D B* are given, we know the angle *A D B*; from the given velocities, we also know the lines *D A* and *D B*, described in a certain time. Calculate the side *A B*, and the angles *A* and *B*. Find the place of the centre of gravity *G* between the bodies at *A* and *B*. Then, in the triangle *D B G*, *D B*, *B G*, and angle *B* are known, by which may be found the distance *D G* passed over by the centre of gravity in the time, and *B D G* the angle which its path makes with that of the body *B*.

2. Three bodies of given weight, *A*, *B*, *C*, in the same time and in the same order, describe with uniform velocity the three sides of the given triangle *A B C* (Fig. 55); required the path of their centre of gravity.

FIG. 55.

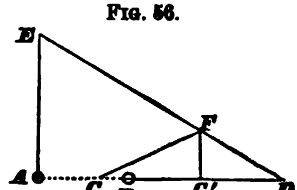


Let *G* be their centre of gravity before they move. If they move successively, *G* describes *G K*, *K L*, *L M*, parallel to the sides of the triangle, and having to them respectively the same ratios as the corresponding moving bodies have to the sum of the bodies (Art. 89). Thus, three sides of the polygon are known; and the angle $K = B$, and $L = C$. These data are sufficient for calculating the fourth side, *G M*, which the centre of gravity describes, when the bodies move together.

3. Show that when the three bodies in Example 2 are equal, the centre of gravity will remain at rest.

4. *A* (Fig. 56) weighs *one* pound; *B* weighs *two* pounds, and lies directly east of *A*; they move simultaneously, *A* northward, and *B* eastward, at the same uniform rate of 40 feet per second; required the direction and velocity of their centre of gravity.

FIG. 56.



Ans. Velocity is 29.814 feet per second.
Direction is E. 26° 33′ 54″ N.

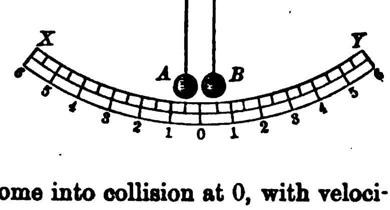
CHAPTER V.

ELASTICITY.

92. Elastic and Inelastic Bodies.—Elastic bodies are those which, when compressed, or in any way altered in form, tend to return to their original state. Those which show no such tendency are called inelastic or non-elastic. No substance is known which is entirely destitute of the property of elasticity; but some have it in so small a degree that they are called inelastic, such as lead and clay. *Elasticity is perfect when the restoring force*, whether great or small, *is equal to the compressing force.* Air, and the gases generally, seem to be almost perfectly elastic; ivory, glass, and tempered steel, are imperfectly, though highly, elastic; and in different substances, the property exists in all conceivable degrees between the above-named limits.

93. Collision.—Mode of Experimenting.—Experiments on collision may be made with balls of the same density suspended by long threads, so as to move in the line which joins their centres of gravity. If the arcs through which they swing are short compared with their radii, the balls, let fall from different heights, will reach the bottom sensibly at the same time, and will impinge *with velocities which are very nearly proportional to the arcs.* Thus A (Fig. 57), falling from 6, and B from 3, will come into collision at 0, with velocities which are as 2 : 1.

Fig. 57.



94. Collision of Inelastic Bodies.—Such bodies, after impact, move together as one mass.

The velocity of two inelastic bodies after collision is equal to the algebraic sum of their momenta, divided by the sum of their masses.

Let A, B, represent the masses of the two bodies, and a, b, their respective velocities. Considering a as positive, if B moves

in the opposite direction, its velocity must be called $-b$. Let v be the common velocity after impact, and suppose the bodies to be moving in the same direction, the momentum of A is $A\,a$; that of B is $B\,b$; and the momentum of both after collision is $(A + B)\,v$. According to the third law of motion (Art. 13), whatever momentum A loses, B gains, so that the whole momentum is the same after collision as before; therefore

$$A\,a + B\,b = (A + B)\,v\,; \therefore v = \frac{A\,a + B\,b}{A + B}.$$

To find the gain or loss of velocity of either body subtract the resulting velocity from the original velocity; a negative result indicates motion in a direction opposite to the original motion.

95. Questions on Inelastic Bodies.—

1. A, weighing 3 oz., and moving 10 feet per second, overtakes B, weighing 2 oz., and moving 3 feet per second; what is the common velocity after impact? *Ans.* $7\frac{1}{5}$ feet per second.

2. A weight of 7 oz., moving 11 feet per second, strikes upon another at rest weighing 15 oz.; required the velocity after impact? *Ans.* $3\frac{1}{2}$ feet per second.

3. A weighs 4 and B 2 pounds; they meet in opposite directions, A with a velocity of 9, and B with one of 5 feet per second; what is the common velocity after impact?
Ans. $4\frac{1}{3}$ feet per second.

4. $A = 7$ pounds, $B = 4$ pounds; they move in the same direction, with velocities of 9 and 2 feet per second; required the velocity lost by A and gained by B? *Ans.* A $2\frac{6}{11}$, B $4\frac{5}{11}$.

5. A body moving 7 feet per second, meets another moving 3 feet per second, and thus loses half its momentum; what are the relative masses of the two bodies?
Ans. $A : B :: 13 : 7$.

6. A weighs 6 pounds and B 5; B is moving 7 feet per second in the same direction as A; by collision B's velocity is doubled; what was A's velocity before impact?
Ans. $19\frac{5}{6}$ feet per second.

7. A body weighing 100 lbs., and having velocity 40 feet per second meets another weighing 20 lbs., and having velocity of 200 feet per second; what will be the velocity after impact?
Ans. 0.

96. Collision of Elastic Bodies.—Elastic bodies after collision do not move together, but each has its own velocity. These velocities are found by *doubling* the *loss* and *gain* of inelastic bodies. When the elastic body A impinges on B, it loses velocity

while it is becoming compressed, and again, while recovering its form, it loses as much more, because the restoring force is equal to the compressing force. For a like reason, B gains as much velocity while recovering its form as it gained while being compressed by the action of A. Hence, doubling the expressions for loss and gain found by Art. 94, and applying them to the original velocities, we find the velocity of each body after collision, on the supposition of perfect elasticity.

Two equal elastic bodies, A and B, weighing 50 lbs. each, moving with velocities, $A = 40$ ft., and $B = 20$ ft. per second, meet; what will be the velocity of each after impact? First we must find the gain and loss of velocity on the supposition that the bodies are inelastic, and then double such gain or loss; therefore, according to Art. 94, calling the velocity after impact v, we have $(50 + 50)\,v = 50 \times 40 - 50 \times 20$, calling the velocity of B negative as the bodies move in opposite directions,—

$$\text{whence } v = \frac{1000}{100} = 10.$$

A loses $40 - 10 = 30$ ft. per second, and B loses $-20 - 10 = -30$ feet per second; that is to say, B loses all its motion in its original direction, and moves backward with velocity 10.

Now as these are elastic we must double the gain and loss, and we have A's loss $= 60$ ft. and B's $= -60$ ft.; therefore A must move with velocity $40 - 60 = -20$, and B with velocity $-20 - (-60) = 40$, hence A must now move in a direction opposite to the first, with velocity 20, and B also in direction opposite to its previous motion, with velocity 40. *Each body takes the velocity of the other when the bodies are equal.*

97. Questions on Elastic Bodies.—

1. A, weighing 10 lbs. and moving 8 feet per second, impinges on B, weighing 6 lbs. and moving in the same direction, 5 feet per second; what are the velocities of A and B after impact?
Ans. A's $= 5\frac{3}{4}$, B's $= 8\frac{3}{4}$.

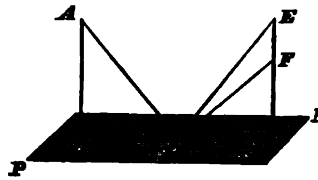
2. $A : B :: 4 : 3$; directions the same; velocities $5 : 4$; what is the ratio of their velocities after impact? *Ans.* $29 : 36$.

3. A, weighing 4 lbs., velocity 6, meets B, weighing 8 lbs., velocity 4; required their respective directions and velocities after collision? *Ans.* A is reflected back with a velocity of $7\frac{1}{3}$, and B with a velocity of $2\frac{2}{3}$.

4. A and B move in opposite directions; A equals $4\,B$, and $b = 2\,a$; how do the bodies move after collision?
Ans. A returns with $\frac{1}{5}$, B with $1\frac{2}{5}$ its original velocity.

98. Impact on an Immovable Plane.—If an inelastic body strikes a plane perpendicularly, its velocity is simply *destroyed;* its energy, however, is transformed into heat. If it strikes obliquely, and the plane is smooth, it slides along the plane with a diminished velocity. Let $A\ L$ (Fig. 58) represent the velocity of the body before impact on the plane $P\ N$, and resolve it into $A\ C$, perpendicular, and $C\ L$, parallel to the plane. Then $A\ C$, as before, is destroyed, but $C\ L$ is not affected; hence its velocity on the plane equals its former velocity times the cosine of the inclination.

FIG. 58.



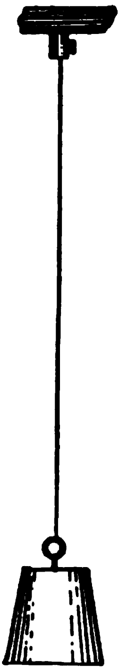
If a perfectly elastic body impinges perpendicularly upon a plane, then, after its velocity is destroyed, the force by which it resumes its form causes an equal velocity in the opposite direction; that is, the body rebounds in its own path as swiftly as it struck. But if the impact is oblique, the body rebounds at an equal angle on the opposite side of the perpendicular. For, resolve $A\ L$, as before, into $A\ C$, $C\ L$; the latter continues uniformly; but, instead of the component $A\ C$, there is an equal motion in the opposite direction. Therefore, if $L\ D$ is made equal to $C\ L$, and $D\ E$ equal to $A\ C$, the resultant of $L\ D$ and $D\ E$ is $L\ E$, which is equal to $A\ L$, and has the same inclination to the plane. Hence, the angles of incidence and reflection are equal, and on opposite sides of the perpendicular to the surface at the place of impact.

99. Imperfect Elasticity.—The formulæ for the velocity of bodies after collision, and the statements of the preceding article, are correct only on the supposition that bodies are, on the one hand, entirely destitute of elasticity, or on the other perfectly elastic. As no solid bodies are known, which are strictly of either class, these deductions are found to be only near approximations to the results of experiment. In all practical cases of the impact of movable bodies, the loss and gain of velocity are *greater* than if they were inelastic, and *less* than if perfectly elastic. And in cases of impact on a plane, there is always *some* velocity of rebound, but less than the previous velocity; and therefore, if the collision is oblique, the body has less velocity, and makes a smaller angle with the plane than before. For, making $D\ F$ less than $A\ C$, the resultant $L\ F$ is less than $A\ L$, and the angle $D\ L\ F$ is smaller than $D\ L\ E$, or $A\ L\ C$.

The kinetic energy of colliding elastic bodies is preserved after impact. Inelastic bodies lose a portion, which appears as heat.

100. Elasticity of Traction.—Bodies, in the form of bars or wires, when fastened vertically at their upper ends, and with weights applied at their lower ends, suffer longitudinal expansion (Fig. 59). Upon removing the weights the bodies resume their former dimensions in virtue of their elasticity, providing the weights are not too great. If, after adding successively heavier weights, a body fails to return to its former dimensions, it is said to have been stretched beyond the *limits of elasticity.*

FIG 59.



Experiments show that (within the limits of elasticity) for elasticity of traction

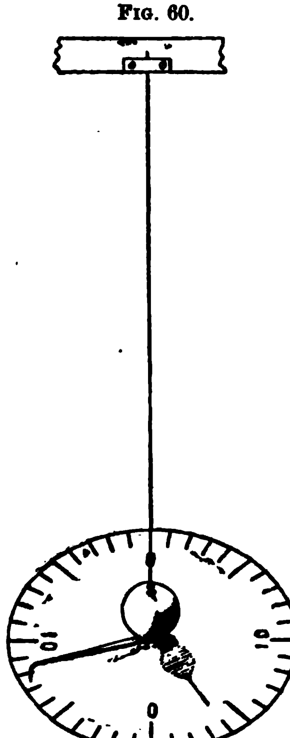
The alteration in length is in proportion to the total length and the acting load, and is inversely as the cross-section of the body.

It is also dependent upon a constant pertaining to the substance of the body, termed the *modulus of elasticity.* The relations are best seen by employing a formula. Let

L = length of body;
l = increase of length by load;
P = load in kg.;
Q = cross-section in sq. mm.;
μ = modulus.

Then $$l = \frac{1}{\mu}\frac{P\,L}{Q}, \text{ whence } \mu = \frac{P\,L}{l\,Q}.$$

FIG. 60.



The modulus of elasticity is the number of kilograms of load required to double the length of a body, if its cross-section be one square millimeter.

Some substances, when strained or compressed, exhibit a property termed *elastic fatigue.* The alteration of length increases somewhat, if the load be allowed to act for some time. This then would give two moduli. In technical tables the modulus for instantaneous elongation is given. After continued load, the same bodies exhibit fatigue in returning to their former shape.

101. Elasticity of Torsion.—If a weight, suspended by a wire (Fig. 60)

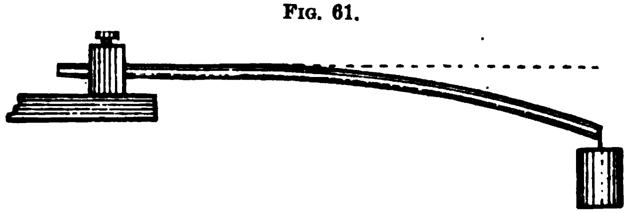
and supplied with an index moving over a graduated circle, be twisted through an angle and then released, the torsional elasticity of the wire will cause it to turn back into its former position. The inertia of the weight will cause a twist in an opposite direction and the index will continue to describe a series of very nearly isochronous oscillations about the point of equilibrium.

Experiments upon elasticity of torsion have shown that *the amount of force exerted by a twisted wire is directly proportional to the angle through which it has been twisted.*

This fact is of great value, as it lies at the foundation of the construction of many instruments for exact physical measurement.

102. Elasticity of Flexure.—A substance, in the form of a bar, clamped at one end, so as to lie in a horizontal position, will be deflected (Fig. 61) from the horizontal if a weight be ap-

FIG. 61.



plied at the free end. The amount of the deflection of the free end,

$$S = \frac{4\,P\,l^3}{\mu\,a^3\,b},$$

where l = length of free bar, P = applied weight, μ = modulus of elasticity, a = vertical side, and b = horizontal breadth.

Elasticity of flexure finds many practical applications. Watch-springs, carriage springs, spring balances, dynamometers, all spiral springs are dependent upon elasticity of flexure for their action.

CHAPTER VI.

SIMPLE MACHINES.

103. Classification of Machines.—In the preceding chapters the motion of bodies has been supposed to arise from the immediate action of one or more forces. But a force may produce effects *indirectly*, by means of something which is interposed for the purpose of changing the mode of action. These intervening bodies are called, in general, *machines;* though the names, *tools, instruments, engines,* &c., are used to designate particular classes of them. The elements of machinery are called *simple machines.* The following list embraces those in most common use :

1. The lever.
2. The wheel and axle.
3. The pulley.
4. The rope machine.
5. The inclined plane.
6. The wedge.
7. The screw.
8. The knee-joint.

In respect to principle, these eight, and all others, may be reduced to three.

1. The law of *equal moments*, applicable in those cases in which the machine turns on a pivot or axis, as in the lever and the wheel and axle.

2. The principle of *transmitted tension*, to be applied wherever the force is exerted through a flexible cord, as in the pulley or rope machine.

3. The principle of *oblique action*, applicable to all the other machines, the force being employed to balance or overcome one component only of the resistance.

The force which transmits its energy to the machine is called the *power*. This word is not to be confused with power as defined in Art. 38. From long usage by various text-book authors, the word has become so firmly attached to machines that it does not seem advisable to discard it.

The force which resists the power, and is balanced or overcome by it, is called the *weight*.

A *compound* machine is one in which two or more simple machines are so connected that the weight of the first constitutes the power of the second, the weight of the second the power of the third, &c.

I. The Lever.

104. The Three Orders of Straight Lever.—The lever is a bar of any form, free to turn on a fixed point, which is called the *fulcrum.* In the *first* order of lever, the *fulcrum* is between the power and weight (Fig. 62); in the *second,* the *weight* is between the power and fulcrum (Fig. 63); in the *third,* the *power* is between the weight and fulcrum (Fig. 64).

Fig. 62.

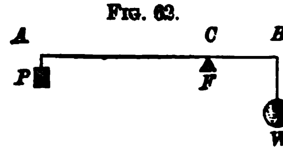


Fig. 63.

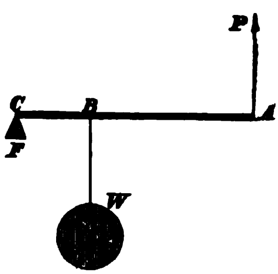
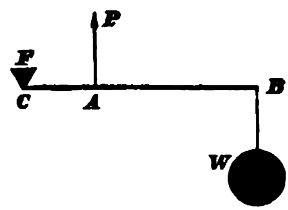


Fig. 64.



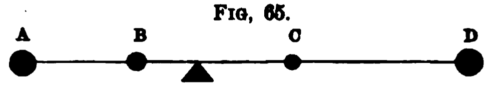
105. Equal Moments in Relation to the Fulcrum.—According to the principle of moments, we find for each order of the lever, $P \times A\,C = W \times B\,C$; that is,

The power and weight have equal moments in relation to the fulcrum.

The *moment* of either force is the measure of its efficiency to turn the lever; for, since the lever is in equilibrium, the efficiency of the power to turn it in one direction must equal the efficiency of the weight to turn it in the opposite direction. We may therefore use $P \times A\,C$ to represent the former, and $W \times B\,C$, the latter.

If *several* forces, as in Fig. 65, are in equilibrium, some tending

Fig. 65.



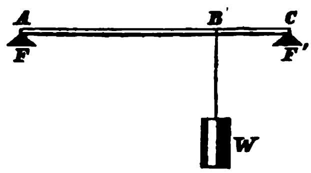
to turn the bar in one direction, and others in the opposite, then A and B must have the same efficiency to produce one motion as C and D have to produce the opposite; that is, $A \times A\,G + B \times B\,G = C \times C\,G + D \times D\,G$; or,

The *sum of the moments* of A and B equals the *sum of the moments* of C and D.

In order to allow for the influence of the weight of the lever itself, consider it to be collected at its centre of gravity, and add its moment to that of the power or weight, according as it aids the one or the other. In Fig. 62, let the weight of the lever $= w$, and the distance of its centre from C on the side of $P = m$; then $P \times A\,C + m\,w = W \times B\,C$. In the 2d and 3d orders, the moment of the lever necessarily aids the weight; and hence, in each case, $P \times A\,C = W \times B\,C + m\,w$.

If a weight hangs on a bar between two supports, as in Fig. 66, it may be regarded as a lever of the 2d order, the reaction of either support being considered as a power. Let F denote the reaction at A, and F' at C; then by the theorems of parallel forces, we have the pressures at A and C inversely as their distances from B, and $W = F + F'$.

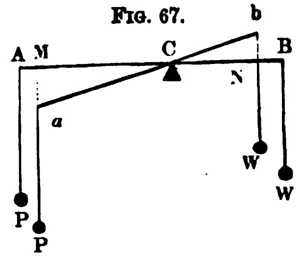
Fig. 66.



Resuming the equation $P \times A\,C = W \times B\,C$, we derive the proportion $P : W :: B\,C : A\,C$; hence, in each order of the straight lever, when the forces act in parallel lines, *The power and weight are inversely as the lengths of the arms on which they act.*

106. The Acting Distance.—In the three orders, as above described, the equilibrium is not destroyed by inclining the lever to any angle whatever with the horizon, provided the lever is symmetrical with respect to its longer axis and the centre of motion C is on this axis and not *above* or *below* it, and provided the directions of the forces remain vertical. For by the principle of parallel forces *any* straight line intersecting the lines of the forces is divided by the line of the resultant into parts which are inversely as the forces; therefore (Fig. 67) $b\,C : a\,C :: P : W$. Hence, the resultant of P and W remains at C, in every position of the lever. By similar triangles, $b\,C : a\,C :: C\,N : C\,M$; $\therefore P : W :: C\,N : C\,M$; $\therefore P \times C\,M = W \times C\,N$. The lines $C\,M$ and $C\,N$, which are drawn from the fulcrum perpendicular to the lines in which the forces act, are called the *acting distances* or the *lever arms* of the power and weight, respectively. And as they may be employed in levers of irregular form, the moments of power and weight are usually measured by the products, $P \times C\,M$ and $W \times C\,N$; therefore, *the power multiplied*

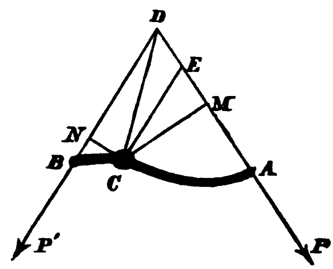
Fig. 67.



by its acting distance equals the weight multiplied by its acting distance; or, more briefly, the moment of the power equals the moment of the weight, as in Art. 105. In Figs. 62, 63, and 64, the acting distances are in each case identical with the arms of the lever.

107. Lever not Straight, and Forces not Parallel.—Let $A\ C\ B$ (Fig. 68) be a lever of any form, and let it be in equilibrium by the forces P and P', acting in any oblique directions in the same plane. Produce $P\ A$ and $P'\ B$ till they meet in D; then, if the fulcrum is at C, the resultant must be in the direction $D\ C$; otherwise the reaction of the fulcrum cannot keep the system in equilibrium (Art. 43). Therefore (Art 44),

FIG. 68.



$$P : P' :: \sin B\ D\ C : \sin A\ D\ C.$$

Draw $C\ M$ perpendicular to $A\ D$, and $C\ N$ to $B\ D$, and they are the sines of $A\ D\ C$ and $B\ D\ C$, to the same radius $D\ C$.

$$\therefore\ P : P' :: C\ N : C\ M; \text{ and } P \times C\ M = P' \times C\ N.$$

The lines $C\ M$ and $C\ N$ are the acting distances of P and P'; therefore the law of the lever in all cases is the same, namely:

The moment of the power equals the moment of the weight.

When the forces act obliquely, the pressure on the fulcrum is less than the sum of the forces; for, if $C\ E$ is parallel to $B\ D$, then $D\ E$, $E\ C$, and $C\ D$, represent the three forces which are in equilibrium. But $C\ D$ is less than the sum of $D\ E$ and $E\ C$.

108. The Compound Lever.—When a lever acts on a second, that on a third, &c., the machine is called a *compound lever*. The law of equilibrium is—

The continued product of the power and acting distances on the side of the power is equal to the continued product of the weight and acting distances on the side of the weight.

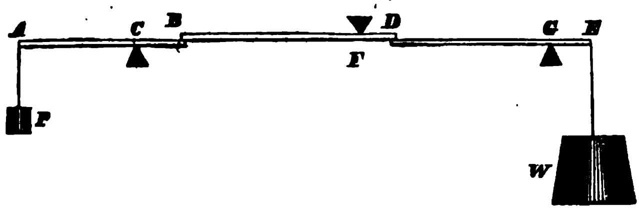
Let the force exerted by $A\ B$ on $B\ D$ (Fig. 69) be called x, and that of $B\ D$ on $D\ E$ be called y; then

$$P \times A\ C = x \times B\ C;$$
$$x \times B\ F = y \times D\ F;$$
$$y \times D\ G = W \times G\ E.$$

Multiply these equations together and omit common factors, and we have

$$P \times AC \times BF \times DG = W \times BC \times DF \times GE.$$

FIG. 69.



If the levers were of irregular forms, the acting distances might not be identical with the arms, as they are in the figure.

109. The Balance.—This is a common and valuable instrument for weighing. It is a straight lever with equal arms, having scale-pans, either suspended at the ends, or standing upon them, one to contain the poises, and the other the substance to be weighed. For scientific purposes, particularly for chemical analysis, great care is bestowed on the construction of the balance.

The arms of the balance, measured from the fulcrum to the points of suspension, must be precisely equal.

The knife-edges forming the fulcrum, and the points of suspension, are made of hardened steel, and arranged exactly in a straight line.

The centre of gravity of the beam is *below* the fulcrum, so that there may be a stable equilibrium; and yet below it by an exceedingly small distance, in order that the balance may be very sensitive.

To preserve the edge of the fulcrum from injury, the beam is raised by supports called *Y's*, when not in use.

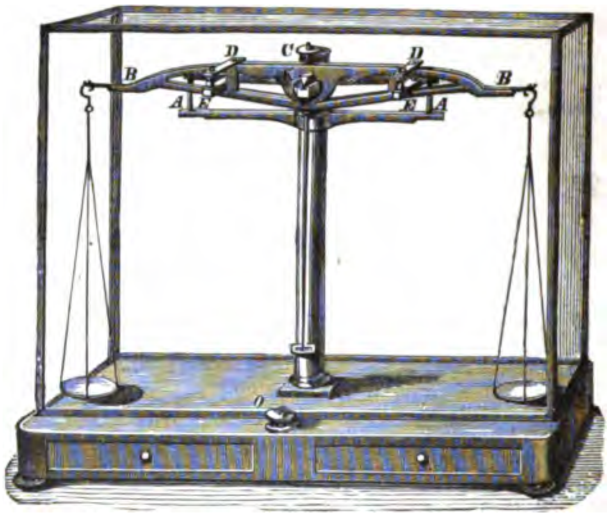
A long index at right angles to the beam, points to zero on a scale when the beam is horizontal.

To protect the instrument from dust and moisture at all times, and from air-currents while weighing, the balance is in a glass case, whose front can be raised or lowered at pleasure.

A balance for chemical analysis is shown in Fig. 70. By turning the knob *O*, the beam can be raised on the *Y's A A* from the surface on which the fulcrum *K* rests. The screw *C* raises and lowers the fulcrum in relation to the centre of gravity of the beam, in order to increase or diminish the sensitiveness of the instrument. In the most carefully made balances, the index will make

a perceptible change, by adding to the scale *one* millionth of the poise.

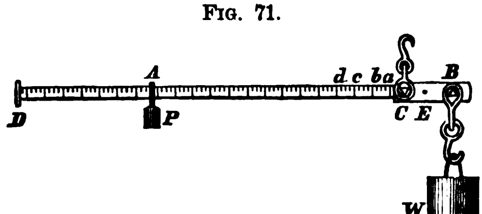
FIG. 70.



For commercial purposes, it is convenient to have the scale-pans above the beam. This is done by the use of additional bars, which with the beam form parallelograms, whose upright sides are rods, projecting upward and supporting the scales. Such contrivances necessarily increase friction ; but balances so constructed are sufficiently sensitive for ordinary weighing.

110. The Steelyard.—This is a weighing instrument, having a graduated arm, along which a poise may be moved, in order to balance various weights on the short arm. While the moment of the article weighed is changed by increasing or diminishing its quantity, that of the poise is changed by altering its acting distance. Since $P \times A\ C = W \times B\ C$ (Fig. 71), and P is constant,

FIG. 71.



and also the distance $B\ C$ constant, $A\ C \propto W$; hence, if W is successively 1 lb., 2 lbs., 3 lbs., &c., the distances of the notches,

a, b, c, &c., are as 1, 2, 3, &c.; in other words, the bar $C D$ is divided into equal parts. In this case, the graduation begins from the fulcrum C as the zero point.

But suppose, what is often true, that the centre of gravity of the steelyard is on the long arm, and that P placed at E would balance it; then the moment of the instrument itself is on the side $C D$, and equals $P \times C E$. Hence, the equation becomes

$$P \times A C + P \times C E = W \times B C; \text{ or}$$
$$P \times A E = W \times B C.$$

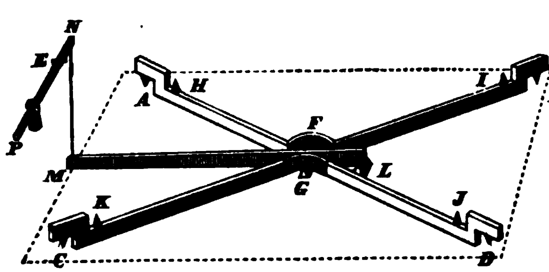
$\therefore W \propto A E$; and the graduation must be considered as commencing at E for the zero point. Such a steelyard cannot weigh below a certain limit, corresponding to the first notch a.

To find the length of the divisions on the bar, divide $A E$, the distance of the poise from the zero point, by W, the number of units balanced by P, when at that distance.

The steelyard often has *two* fulcrums, one for less and the other for greater weights.

111. Platform Scales.—This name is given to machines arranged for weighing heavy and bulky articles of merchandise. The largest, for cattle, loaded wagons, &c., are constructed with the platform at the surface of the ground. In order that the platform may stand firmly beneath its load, it rests by four feet on as many levers of the second order, whose arms have equal ratios. $A F, B F, C G, D G$ (Fig. 72), are four such levers, resting on the

FIG. 72.



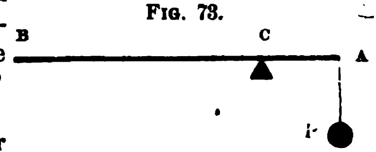
fulcrums, A, B, C, D, while the other ends meet on the knife-edge, $F G$, of another lever, $L M$. This fifth lever has its fulcrum at L, and its outer extremity is attached by a vertical rod, $M N$, to a steelyard, whose fulcrum is E, and poise P. The five levers are arranged in a square cavity just below the surface of the ground. The dotted line shows the outline of the cavity. On the bearing points of the four levers, H, I, J, K, rest the feet of the platform (not represented), which is firmly built of plank, and just

fits into the top of the cavity without touching the sides. The machine is a compound lever of three parts; for the four levers act as one at $F\ G$, and are used to give steadiness to the platform which rests upon them.

A construction quite similar to the above is made of portable size, and used in all mercantile establishments for weighing heavy goods.

112. Questions on the Lever.—1. $A\ B$ (Fig. 73) is a uniform bar, 2 feet long, and weighs 4 oz.; where must the fulcrum be put, that the bar may be balanced by P, weighing 5 lbs.?

FIG. 73.



Ans. $\frac{4}{7}$ of an inch from A.

2. A lever of the second order is 25 feet long; at what distance from the fulcrum must a weight of 125 pounds be placed, so that it may be supported by a power able to sustain 60 pounds, acting at the extremity of the lever.

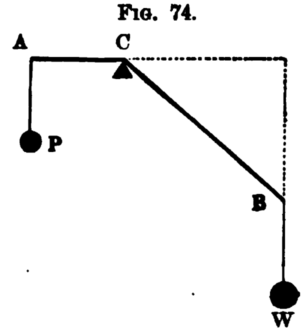
Ans. 12 feet.

3. A and B are of the same height, and sustain upon their shoulders a weight of 150 pounds, placed on a pole $9\frac{1}{3}$ feet long; the weight is placed $6\frac{2}{3}$ feet from A; what is the weight sustained by each person?

Ans. A sustains $42\frac{6}{7}$ lbs., and B sustains $107\frac{1}{7}$ lbs.

4. A *bent lever*, $A\ C\ B$ (Fig. 74), has the arm $A\ C = 3$ feet, $C\ B = 8$ feet, $P = 5$ lbs., and the angle $A\ C\ B = 140°$; what weight, W, must be attached at B, in order to keep $A\ C$ horizontal?

FIG. 74.



Ans. 2.4476 lbs.

5. A cylindrical straight lever is 14 feet long, and weighs 6 lbs. 5 oz.; its longer arm is 9, and its shorter 5 feet; at the extremity of its shorter arm a weight of 15 lbs. 2 oz. is suspended; what weight must be placed at the extremity of the longer arm to keep it in equilibrium?

Ans. 7 lbs.

6. A uniform bar, 12 feet long, weighs 7 lbs.; a weight of 10 lbs. hangs on one end, and 2 feet from it is applied an upward force of 25 lbs., where must the fulcrum be put to produce equilibrium?

Ans. 1 foot from the 10 lbs.

7. The lengths of the arms of a balance are a and b. When p ounces are hung on a, they balance a certain body; but it re-

quires q ounces to balance the same body, when placed in the other scale. What is the true weight of the body? According to the first weighing, $a\,p = b\,x$; according to the second, $b\,q = a\,x$. $\therefore a\,b\,p\,q := a\,b\,x^2$, and $x = \sqrt{p\,q}$. Hence, the true weight is a geometrical mean between the apparent weights.

8. On one arm of a false balance a body weighs 11 lbs., on the other, 17 lbs. 3 oz.; what is the true weight?

Ans. 13 lbs. 12 oz.

9. Four weights of 1, 3, 5, 7 lbs., respectively, are suspended from points of a straight lever, eight inches apart; how far from the point of suspension of the first weight must the fulcrum be placed, that the weights may be in equilibrium?

Ans. 17 inches.

10. Two weights keep a horizontal lever at rest, the pressure on the fulcrum being 10 lbs., the difference of the weights 4 lbs., and the difference of the lever arms 9 inches; what are the weights and their lever arms?

Ans. Weights, 7 lbs. and 3 lbs.; arms, 6¾ in. and 15¾ in.

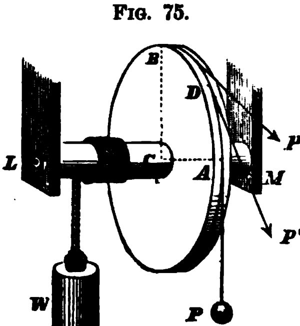
II. The Wheel and Axle.

113. Description and Law of the Machine.—The wheel and axle consists of a cylinder and a wheel, firmly united, and free to revolve on a common axis. The power acts at the circumference of the wheel in the direction of a tangent, and the weight in the same manner, at the circumference of the cylinder or axle; so that the acting distances are the radii at the two points of contact. As the system revolves, the radii successively take the place of acting distances, without altering at all the relation of the forces to each other. The wheel and axle is therefore a kind of endless lever.

Let W (Fig. 75) be the weight suspended from the axle, tending to revolve it on the line $L\,M$; and P, the power acting on the wheel, tending to revolve the system in the opposite direction. It is plain that the acting distances are the radius of the axle, and $A\,C$ the radius of the wheel. In case of equilibrium, the moment of W equals the moment of P. Calling the radius of the axle r, and the radius of the wheel R, then $W \times r = P \times R$; or

$$P : W :: r : R.$$

FIG. 75.

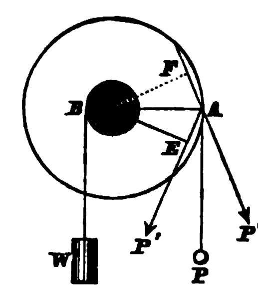


If, instead of the weight P, suspended on the wheel, the rope be drawn by any force in the direction P' or P'', it is still tangent to the circumference, and therefore its acting distance, $C\,D$ or $C\,B$, the same as before. In general, the law of equilibrium for this machine is,

The moment of the Power is equal to the moment of the Weight.

If the rope on the wheel, being fastened at A (Fig. 76) is drawn by the side of the wheel, as $A\ P'$, the acting distance of the power is diminished from $C\,A$ to $C\,E$, and therefore its efficiency is diminished in the same ratio. Were the rope drawn away from the wheel, as $A\ P''$, making an equal angle on the other side of $A\ P$, the same effect is produced, the acting distance now becoming $C\,F$.

FIG. 76.



The radius of the wheel and the radius of the axle should each be reckoned from the axis of rotation to the centre of the rope; that is, half of the thickness of the rope should be added to the radius of the circle on which it is coiled. Calling t the half thickness of the rope on the axle, and t' that of the rope on the wheel, the equation of equilibrium is

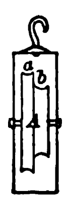
$$P \times (R + t') = W \times (r+t).$$

In considering the wheel and axle no account has been taken of the stiffness of the rope, which acts as a constant resistance, opposing motion in winding upon a drum or wheel, and also in unwinding.

114. Differential Pulley.—A modification of the wheel and axle, called a *differential pulley*, is of great use in raising very heavy weights through short distances.

The pulley consists of a *solid* wheel A (Fig. 77), one half of which, b, is of less diameter than the other half, a, suspended in a block in the usual manner.

FIG. 77.

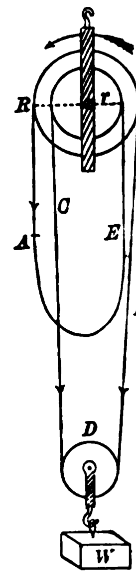


A continuous *chain* is used, which we may trace from the point A (Fig. 78), upward, over the larger of the two circumferences, then downward through B to the movable pulley D, thence upward through C around the smaller circumference of the wheel, thence down through E and back to the point of beginning at A.

Call the radii R and r as indicated in the figure, and suppose

a downward force P to be applied to the chain at A, then $P \times R + \frac{1}{2} W \times r = \frac{1}{2} W \times R$, in which equation no account is taken of the weight of the chain.

FIG. 78.



Transposing, we obtain,

$$P \times R = \tfrac{1}{2} W (R-r) \text{ or}$$
$$P : \tfrac{1}{2} W :: R-r : R.$$

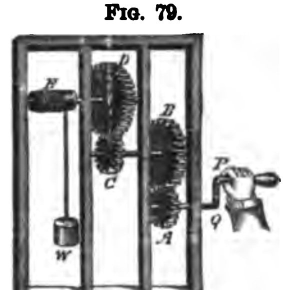
Now $R-r$ may be made as small as we please, and hence the power also may be made small as compared with the weight. The weight of the chain and the friction act as resistances to motion, and are sufficient to prevent the downward run of D after the hand is removed from A, even when W is very great. This pulley may be found in any large foundry, or machine shop.

115. The Compound Wheel and Axle.—When a train of wheels, like that in Fig. 79, is put in motion, those which *communicate* motion by the circumference are called *driving wheels*, as A and C; those which *receive* motion by the circumference are called *driven wheels*. And the law of equilibrium is,

The continued product of the power and radii of the driven wheels is equal to the continued product of the weight and radii of the driving wheels.

The crank $P\,Q$ is to be reckoned among driven wheels; the axle E among driving wheels.

FIG. 79.



Let the radius of B be called R; of D, R'; of A, r; of C, r'; of E, r''. Call the force exerted by A on B, x; that of C on D, y. Then

$$P \times P\,Q = r \times x;$$
$$x \times R = r' \times y;$$
$$y \times R' = r'' \times W.$$

Multiply and omit common factors, and we have

$$P \times P\,Q \times R \times R' = W \times r'' \times r' \times r.$$

If the driving wheels are equal to each other, and also the driven wheels, and the number of each is n, then

$$P \times R^n = W r^n.$$

116. Direction and Rate of Revolution.—When two wheels are geared together by teeth, they necessarily revolve in contrary directions. Hence, in a train of wheels, the alternate axles revolve the same way.

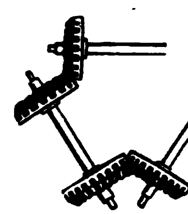
The circumferences of two wheels which are in gear move with

the same velocity; hence the number of revolutions will be reciprocally as the radii of the wheels.

Since teeth which gear together are of the same size, the relative ***number of teeth*** is a measure of the relative circumferences, and therefore of the relative radii of the wheels. If the wheel A (Fig. 79) has 20 teeth, and B has 40, and again if C has 15, and D 45, then for every revolution of B, A revolves twice, and for every revolution of D, C revolves three times. Therefore, six turns of the crank are necessary to give one revolution to the axle E.

By cutting the teeth of wheels on a conical instead of a cylindrical surface, the axles may be placed at any angle with each other, as represented in Fig. 80.

FIG. 80.



Whether axles are parallel or not, ***bands*** instead of teeth may be used for transmitting rotary motion. But as bands are liable to slip more or less, they cannot be employed in cases requiring exact relations of velocity.

117. Questions on the Wheel and Axle.—

1. A power of 12 lbs. balances a weight of 100 lbs. by a wheel and axle; the radius of the axle is 6 inches; what is the *diameter* of the wheel?

Ans. 8 ft. 4 in.

2. $W = 500$ lbs.; $R = 4$ ft.; $r = 8$ in.; the weight hangs by a rope 1 inch thick, but the power acts at the circumference of the wheel without a rope; what power will sustain the weight?

Ans. 88.54 lbs.

3. In Fig. 79, A and C have each 15 teeth, B and D each 40 teeth; the radius of the axle E is 4 inches; the rope on it 1 inch in diameter; and the radius of the crank $P\,Q$ is 18 inches; what is the ratio of power to weight in equilibrium? *Ans.* 1 : $28\frac{4}{9}$.

118. The Pulley Described.—

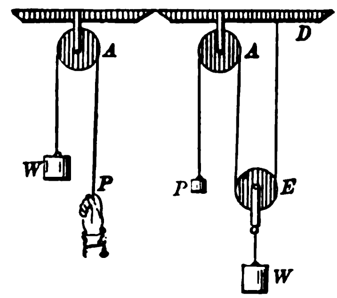
The pulley consists of one or more wheels or rollers, with a rope passing over the edge in which a groove is sunk to keep the rope in place. The axis of the roller is in a *block*, which is sometimes fixed, and sometimes rises and falls with the weight; and the pulley is accordingly called a *fixed pulley* or a *movable pulley*. The principle which explains the relation of power and weight in every form of pulley is this:

Whatever strain or tension is applied to one end of a cord, is transmitted through its whole length, if it does not branch, however much its direction is changed.

In the pulley, the sustaining portions of the rope are assumed to be parallel to each other.

119. The Fixed Pulley.—In the fixed pulley, A (Fig. 81), the force P, produces a tension in the string, which is transmitted through its whole length, and which can be balanced only when W equals P. Hence, in the fixed pulley, *the power and weight are equal.* This machine is useful for changing the *direction* in which the force is applied to the weight; and if the power only acts in the plane of the groove of the wheel, it is immaterial what is its direction, horizontal, vertical, or oblique.

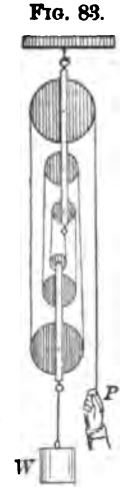
FIG. 81. FIG. 82.



120. The Movable Pulley.—In Fig. 82, the tension produced by P, is transmitted from A down to the wheel E, and thence up to D; therefore W is sustained by *two* portions of the rope, each of which exerts a force equal to P.

$$\therefore W = 2P; \text{ or } P : W :: 1 : 2.$$

FIG. 83.



The same reasoning applies, where the rope passes between the upper and lower blocks any number of times, as in Fig. 83. The force causes a tension in the rope, which is transmitted to every portion of it. If n is the number of portions which sustain the lower block, then W is upheld by nP; and if there is equilibrium, $P : W :: 1 : n$. In the figure, the weight equals *six times* the power. The law of equilibrium, therefore, for the movable pulley with one rope, is this,

The power is to the weight as one to the number of the sustaining portions of the rope.

121. The Compound Pulley.—Wherever a system of pulleys has separate ropes the machine is to be regarded as compound, and its efficiency is calculated accordingly. Figures 84 and 85 are examples. In Fig. 84 call the weight sustained by F, x, and that sustained by D, y. Then (Art. 120),

$$P : x :: 1 : 2;$$
$$x : y :: 1 : 2:$$
$$y : W :: 1 : 2.$$
$$\therefore P : W :: 1 : 2^3 :: 1 : 8.$$

And if n is the number of ropes, $P : W :: 1 : 2^n$.

In Fig. 85 the tension P is transmitted over A directly to the weight at G; the wheel A is loaded, therefore, with $2\,P$, and a tension of $2\,P$ comes upon the second rope, which is transmitted over B to the weight at F. In like manner, a tension of $4\,P$ is transmitted over C to E. The sum of all these being applied to the weight, it must therefore be equal to that sum in case of equilibrium. Therefore, $P : W :: 1 : 1 + 2 + 4 +$ &c. Now the sum of this geometrical series to n terms is $2^n - 1$, $\therefore P : W :: 1 : 2^n - 1$. This combination is therefore a little less efficient than the preceding.

Fig. 84.

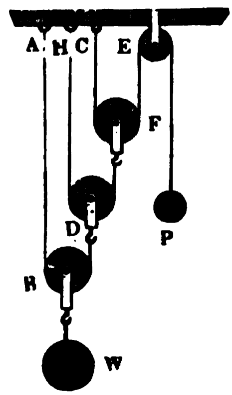
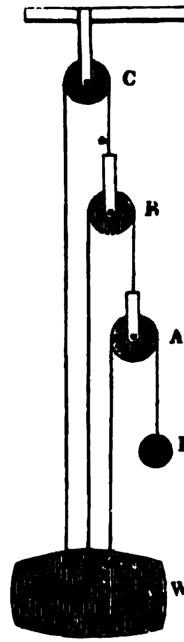


Fig. 85.



Since the several ropes have different tensions, the weight cannot be balanced upon them, unless those of greatest tension are nearest the line of direction of the body. For example, if the rope F is directed toward the centre of gravity of the weight, the rope G should be attached four times as far from it as the rope E, in order to prevent the weight from tipping.

The pulley owes its efficiency as a machine to the fact, that the tension produced by the power is applied *repeatedly* to the weight. The only use of the wheels is to diminish friction. Were it not for friction, the rope might pass round fixed pins in the blocks, and the ratio of power to weight would still be in every case the same as has been shown.

IV. The Rope Machine.

122. Definition and Law of this Machine.—

The rope machine is one in which the power and weight are in equilibrium by the tension of one or more ropes.

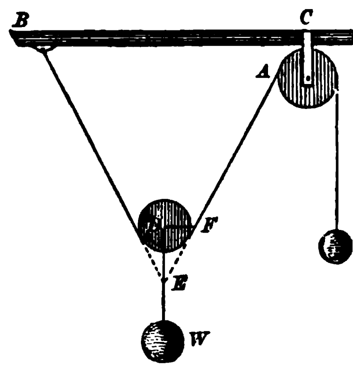
According to this definition the pulley is included. It is that particular form of the rope machine in which the sustaining parts of the ropes are parallel; and it is treated as a separate machine,

because its theory is very simple, and because it is used far more extensively than any other forms.

If the two portions of rope which sustain the weight are inclined, as in Fig. 86, then W is no longer equal to the sum of their tensions, as it is in the pulley, but is always less than that, according to the following law:

The power is to the weight as the sine of ½ the angle is to the sine of the whole angle between the parts of the rope.

FIG. 86.



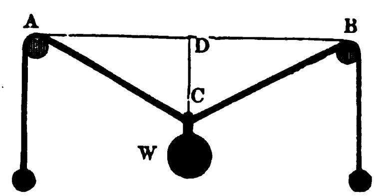
Consider the point E as in equilibrium from three forces— P along $E\,F$, W along $E\,W$ and a force, opposite to their resultant, along $E\,B$. From Art. 44, each force is proportional to the sine of the angle between the other two. Hence,

$$P : W :: (\sin WEB =) \sin BED : (\sin BEF =) \, 2\, BED.$$

123. Change in the Ratio of Power and Weight.— If P is given, all the possible values of W are included between $W = 0$, and $W = 2\,P$.

When the rope is straight from A to B (Fig. 87), so that $C\,D = 0$, then, by the above proportion, $W = 0$. As W is increased from zero, the point C descends; and when $D\,C = \frac{1}{2}\,B\,C$, then, by the proportion, $W = P$. In that case $D\,C\,B = 60°$, and the angles $A\,C\,B$, $A\,C\,W$, and $B\,C\,W$ are equal (each being 120°), as they should be, because each of the equal forces, P, P, and W, is as the sine of the angle between the directions of the other two.

FIG. 87.



But when W has increased to $2\,P$, it descends to an infinite distance; for then, by the proportion, $C\,D = B\,C$, that is, the side of a right-angled triangle is equal to the hypothenuse. Thus, the extreme values of W are 0 and $2\,P$.

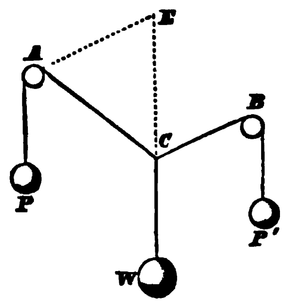
It appears from the foregoing that a perfectly *flexible rope having weight* cannot be drawn into a straight horizontal line by

any force, however great; for C cannot coincide with D, except when $W = 0$.

124. The Branching Rope.—**When C, where the weight is suspended, is a *fixed* point of the rope, we have a branching rope, and the principle of transmitted tension does not apply beyond the point of division.**

Let P, P', and W (Fig. 88), be given, and C a fixed point of the rope. Produce $W C$, and let $A E$, drawn parallel to $C B$, intersect it in E. The sides of $A C E$ are proportional to the given forces; therefore its angles can be found, and the inclinations of $A C$ and $B C$ to the vertical $C W$ are known.

FIG. 88.



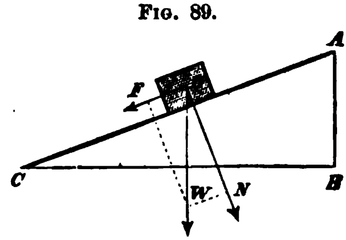
V. THE INCLINED PLANE.

125. Relation of Power, Weight, and Pressure on the Plane.—The mechanical efficiency of the inclined plane is explained on the principle of *oblique action;* that is, it enables us to apply the power to balance or overcome only *one component* of the weight, instead of the whole. Let the weight of the body D, lying on the inclined plane $A C$ (Fig. 89), be represented by W; and resolve it into F parallel, and N perpendicular to the plane. N represents the perpendicular pressure, and is equal to the reaction of the plane; F is the force by which the body tends to move down the plane.

Let a = the angle C, the inclination of the plane; therefore $W D N = a$. Then $F = W$. sin; a and $N = W$. cos a.

Now suppose a force P, in the direction $C A$, applied at D, keeps the body at rest. Evidently

FIG. 89.



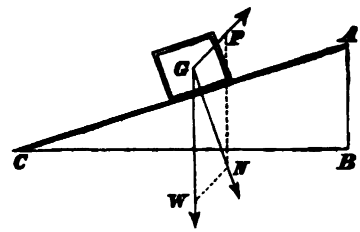
$$P = W \sin a = W \frac{A B}{A C}.$$

Hence, with a force acting parallel to the inclined plane, there is equilibrium when

The power is to the weight as the height to the length of the inclined plane.

If a force P (Fig. 90), having any direction whatever, keeps the body at rest, then the resultant N of P and W must be perpendicular to the plane, for the resultant must have a zero component along any direction which the weight could move, whence $P : W :: (\sin G N P =) \sin a : \sin P G N$.

FIG. 90.



When the force acts in line parallel to the base ($P G N = 90° - a$), we have

$$P : W = \sin a : \cos a = A B : B C, \text{ or}$$

The power is to the weight as the height is to the base of the inclined plane.

126. Power most Efficient when Acting Parallel to the Plane.—From the proportion above

$$W = \frac{P . \sin P G N}{\sin a}.$$

Now, as P and $\sin a$ are given, W varies as $\sin P G N$, which is the greatest possible when $P G N = 90°$; that is, when the power acts in a line parallel to the plane.

Whether the angle $P G N$ diminishes or increases from 90°, its sine diminishes, and becomes zero, when $P G N = 0°$, or 180°. Therefore $W = 0$, or no weight can be sustained, when the power acts in the line $G N$, perpendicular to the plane, either toward the plane or from it.

127. Expression for Perpendicular Pressure.—From the triangle $P G N$ we obtain

$$N : W :: \sin G P N : \sin P G N :: \sin P G W : \sin P G N;$$

$$\therefore N = W \frac{\sin P G W}{\sin P G N}.$$

If the power acts in a line parallel to the inclined plane, $P G W = 90° + a$, $P G N = 90°$, and $N = W \dfrac{\sin (90° + a)}{\sin 90°} = W \cos a$.

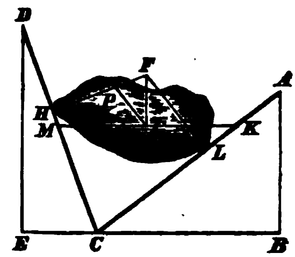
If the power acts in a line parallel to the base of the inclined plane, $P G W = 90°$, $P G N = 90° - a$, and $N = W \dfrac{1}{\cos a} = W \sec a$.

If the power acts in a line perpendicular to the inclined plane,

$$P\,G\,W = a,\ P\,G\,N = 0°,\ \text{and}\ N = W\frac{\sin a}{0} = \infty.$$

128. Equilibrium between Two Inclined Planes.—If a body rests, as represented in Fig. 91, between two inclined planes, the three forces which retain it are its weight, and the resistances of the planes. Draw $H\,F$ and $L\,F$ perpendicular to the planes through the points of contact, and $G\,F$ vertically through the centre of gravity of the body. Since the body is in equilibrium, these three lines will pass through the same point (Art. 43). Let that point be F, and draw $G\,P$ parallel to $L\,F$, and $M\,K$ parallel to the horizon. $G\,P\,F$ is similar to $K\,C\,M$. Therefore (since Pressure on $A\,C$: Pr. on $D\,C$ :: $P\,G : F\,P$),

Fig. 91.



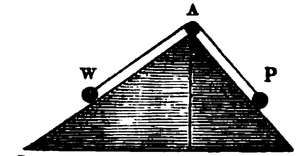
$$\begin{aligned}\text{Pressure on } A\,C : \text{Pr. on } D\,C &:: K\,C : M\,C,\\ &:: \sin M : \sin K,\\ &:: \sin D\,C\,E : \sin A\,C\,B.\end{aligned}$$

That is, when a body rests between two planes, it exerts pressures on them which are inversely as the sines of their inclinations to the horizon.

If, therefore, one of the planes is horizontal, none of the pressure can be exerted on any other plane. *It is friction alone which renders it possible for a body on a horizontal surface to lean against a vertical wall.*

129. Bodies Balanced on Two Planes by a Cord passing over the Ridge.—Let P and W balance each other on the planes $A\,D$ and $A\,C$ (Fig. 92), which have the common height $A\,B$, by means of a cord passing over the fixed pulley A. The tension of the cord is the common power which prevents each body from descending; and as the cord is parallel to each plane, we have (calling the tension t),

Fig. 92.



$$\begin{aligned} t : P &:: A\,B : A\,D;\\ \text{and } t : W &:: A\,B : A\,C;\\ \therefore P : W &:: A\,D : A\,C;\end{aligned}$$

that is, the forces, in case of equilibrium, are directly as the lengths of the planes.

130. Questions on the Inclined Plane.—

1. If a horse is able to raise a weight of 440 lbs. perpendicularly, what weight can he raise on a railway having a slope of five degrees? *Ans.* 5048.5 lbs.

2. The grade of a railroad is 20 feet in a mile; what force must be exerted to sustain any given weight upon it?
Ans. 1 lb. for every 264 lbs.

3. What force is requisite to hold a body on an inclined plane, by pressing perpendicularly against the plane?
Ans. An infinite force.

4. A certain force was able to sustain 500 tons on a plane of $7\frac{1}{2}°$; but on another plane it could sustain only 400 tons; what was the inclination of the latter? *Ans.* 9° 23′ 25″.

5. Equilibrium on an inclined plane is produced when the force, weight, and perpendicular pressure are, respectively, 9, 13, and 6 lbs.; what is the inclination of the plane, and what angle does the force make with the plane?
Ans. $a = 37° \ 21' \ 26''$. Inclination of force to plane $= 28° \ 46' \ 54''$.

6. A force of 10 lbs., acting parallel to the plane, supports a certain weight; but it requires a force of 12 lbs. parallel to the base to support it. What is the weight of the body, and what is the inclination of the plane?
Ans. $W = 18.09$ lbs. $a = 33° \ 33' \ 25''$.

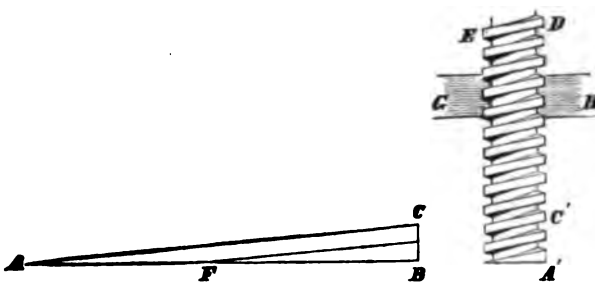
7. To support a weight of 500 lbs. upon an inclined plane of 50° inclination to the horizon, a lifting force is applied whose direction makes an angle of 75° with the horizon. What is the magnitude of this force, and the pressure of the weight against the plane? *Ans.* $P = 422.6$ lbs. $N = 142.8$ lbs.

VI. The Screw.

131. Reducible to the Inclined Plane.—The screw is a cylinder having a spiral ridge or thread around it, which cuts at a constant oblique angle all the lines of the surface parallel to the axis of the cylinder. A hollow cylinder, called a *nut*, having a similar spiral within it, is fitted to move freely upon the thread of the solid cylinder. In Fig. 93, let the base $A\ B$ of the inclined plane $A\ C$ be equal to twice the circumference of the cylinder $A'\ E$; then let the plane be wrapped about the cylinder, bringing the points A, F, and B, to the point A'; then will $A\ C$ describe two revolutions of the thread from A' to C'. Therefore the me-

chanical relations of the screw are the same as of the inclined plane.

Fig. 93.



If a weight be laid on the thread of the screw, and a force be applied to it horizontally in the direction of a tangent to the cylinder, the case is exactly analogous to that of a body moved on an inclined plane by a force parallel to the base. Let r be the radius of the cylinder, then $2 \pi r$ is the circumference; also let d be the distance between the threads, (that is, from any point of one revolution to the corresponding point of the next,) measured parallel to the axis of the cylinder; then $2 \pi r$ is the base of an inclined plane, and d its height. Therefore (Art. 125),

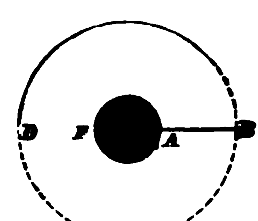
$$P : W :: d : 2 \pi r \text{; or,}$$

The power is to the weight as the distance between the threads measured parallel to the axis, is to the circumference of the screw.

If instead of moving the weight on the thread of the screw, the force is employed to turn the screw itself, while the weight is free to move in a vertical direction, the law is the same. Thus, whether the screw A' E is allowed to rise and fall in the fixed nut G H, or whether the nut rises and falls on the thread of the screw, while the latter is revolved, without moving longitudinally, in each case, $P : W :: d : 2 \pi r$.

132. The Screw and Lever Combined.—The screw is so generally combined with the lever in practical mechanics, that it is important to present the law of the compound machine. Let A F (Fig. 94) be the section of a screw, and suppose B C, a lever of the second order, to be applied to turn it. The fulcrum is at C. the power acts at B, and the effect produced by the lever is at A, the surface of the cylinder. Call that effect x, and let $d =$ the distance between the threads; then,

Fig. 94.



$$P : x :: A\ C : B\ C,$$
$$\text{and } x : W :: d : 2\ \pi\ A\ C;$$

compounding and reducing, we have

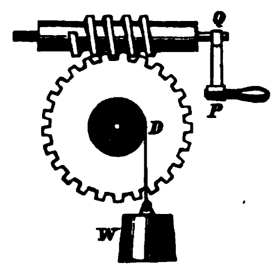
$$P : W :: d : 2\ \pi\ B\ C; \text{ that is,}$$

The power is to the weight as the distance between the threads, measured parallel to the axis, to the circumference described by the power.

The law as thus stated is applicable to the screw when used with the lever or without it.

133. The Endless Screw.—The screw is so called, when its thread moves between the teeth of a wheel, thus causing it to revolve. It is much used for diminishing very greatly the velocity of the weight.

FIG. 95.



Let $P\ Q$ (Fig. 95) be the radius of the crank to which the power is applied; d, the distance between the threads; R, the radius of the wheel; r, the radius of the axle; and call the force exerted by the thread upon the teeth, x. Then,

$$P : x :: d : 2\ \pi \times P\ Q,$$
$$\text{and } x : W :: r : R;$$
$$\therefore P : W :: d\ r : 2\ \pi \times R \times P\ Q.$$

If, for example, $P\ Q = 30$ inches, $d = 1$ in., $R = 18$ in.; $r + t = 2$ in.; then $W = 1696\ P$, and moves with 1696 times less velocity than P.

134. The Right and Left Hand Screw.—The common form of screw is called the *right-hand* screw, and may be described thus: *if the thread in its progress along the length of the cylinder, passes from the left over to the right, it is called a right-hand screw.* Hence, a person in driving a screw forward turns it from his left *over* (not *under*) to his right, and in drawing it back he reverses this movement. Fig. 93 represents a right-hand screw.

The *left-hand* screw is one whose thread is coiled in the opposite direction,—that is, it advances by passing from right over to left. This kind is used only when there is special reason for it. For example, the screws which are cut upon the left-hand ends of carriage axles are left-hand screws; otherwise there would be danger that the friction of the hub against the nut might turn the nut off from the axle. Also, when two pipes for conveying gas or steam are to be drawn together by a nut, one must have a right-hand, and the other a left-hand screw.

135. Questions on the Screw.—

1. The distance between the threads of a screw is one inch, the bar is two feet long from the axis, and the power is 30 lbs.; what is the weight or pressure? *Ans.* 4523.89 lbs.

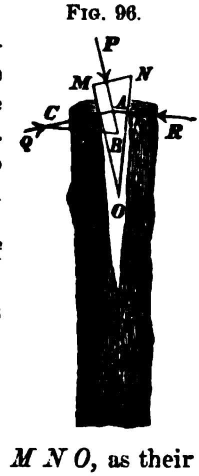
2. The bar is three feet long, reckoned from the axis, $P = 60$ lbs., $W = 2240$ lbs.; what is the distance between the threads? *Ans.* 6.058 inches.

3. A compound machine consists of a crank, an endless screw, a wheel and axle, a pulley, and an inclined plane. The radius of the crank is 18 inches; the distance between the threads of the screw, one inch; the radius of the wheel on which the screw acts, two feet; the radius of the axle, 6 inches; the pulley block has two movable pulleys with one rope, the power exerted by the pulley being parallel to the plane, and the inclination of the plane to the horizon is 30°. What weight on the plane will be balanced by a power of 100 lbs. applied to the crank? *Ans.* 361911.474 lbs.

VII. The Wedge.

136. Definition of the Wedge, and the Mode of Using.—The usual form of the wedge is a triangular prism, two of whose sides meet at a very acute angle. This machine is used to raise a weight by being driven as an inclined plane underneath it, or to separate the parts of a body by being driven between them. When it is used by itself, and does not form part of a compound machine, force is usually applied by a blow, which produces an intense pressure for a short time, sufficient to overcome a great resistance.

Fig. 96.



137. Law of Equilibrium.—Whatever be the direction of the blow or force, we may suppose it to be resolved into two components, one perpendicular to the back of the wedge, and the other parallel to it. The latter produces no effect. The same is true of the resistances; we need to consider only those components of them which are perpendicular to the sides of the wedge.

Let $M N O$ (Fig. 96) represent a section of the wedge perpendicular to its faces; then $P A$, $Q A$, and $R A$, drawn perpendicular to the faces severally, show the directions of the forces which hold the wedge in equilibrium. Taking $A B$ to represent the power, draw $B C$ parallel to $R A$, and we have the triangle $A B C$, whose sides represent these forces. But $A B C$ is similar to $M N O$, as their

sides are respectively perpendicular to each other. Hence, calling the forces P, Q, and R, respectively,

$$P : Q :: MN : MO :$$

$$\text{and } P : R :: MN : NO ;$$

that is, there is equilibrium in a wedge, when

The power is to the resistances as the back of the wedge to the sides on which the resistances respectively act.

If the triangle is isosceles, the two resistances are equal, as the proportions show; and P is to either resistance, R, as the breadth of the back to the length of the side.

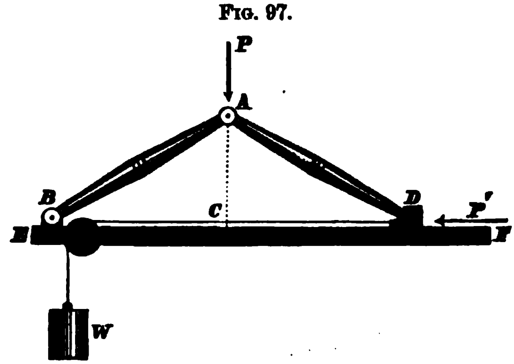
If the resisting surfaces touch the sides of the wedge only in one point each, then $Q\,A$ and $R\,A$, drawn through the points of contact, must meet $A\,P$ in the same point (Art. 43); otherwise the wedge will roll, till one face rests against the resisting body in two or more points.

The efficiency of the wedge is usually very much increased by combining its own action with that of the lever, since the point where it acts generally lies at a distance from the point where the effect is to be produced. Thus, in splitting a log of wood, the resistance to be overcome is the cohesion of the fibers; and this force is exerted at a distance from the wedge, while the fulcrum is a little further forward in the solid wood.

VIII. The Knee-Joint.

138. Description and Law of Equilibrium.—The knee-joint consists of two bars, usually equal, hinged together at one end, while the others are at liberty to separate in a straight line. The power is applied at the hinge, tending to thrust the bars into a straight line; the weight is the force which opposes the separation.

Fig. 97.

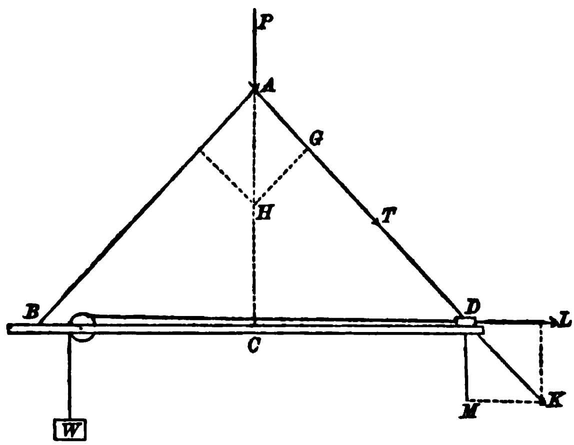


Suppose that $A\,B$ and $A\,D$ (Fig. 97) are equal bars, hinged together at A; and that the bar $A\,B$ is free only to revolve about

the axis B, while the end D of the other bar can move parallel to the base $E F$. If P urges A toward the base, it tends to move D further from the fixed point B. The force P', which opposes that motion, is represented in the figure by the weight W. The law of equilibrium is,

The power is to the weight as twice the height of the joint to half the distance between the ends of the bars.

FIG. 98.



Resolving the force P in the direction of $A\ B$ and $A\ D$, we have, Fig. 98,

$$P : T :: A\ H : A\ G :: 2\ A\ C : A\ D,$$

in which T stands for the component of P in the direction $A\ G$, called the *thrust*.

This component T acts at D and must be again resolved in the directions $D\ L$ and $D\ M$, of which $D\ L$ is equal and opposed to W, and $D\ M$ is equal and opposed to the upward resistance of the plane on which the block D slides, giving the proportion

$$T : W :: D\ K : D\ L \text{ or } M\ K :: A\ D : C\ D.$$

Multiplying like terms of the two proportions and omitting common factors, we have,

$$P : W :: 2\ A\ C : C\ D.$$

139. Ratio of Power and Weight Variable.—It is obvious that the ratio between power and weight is different for different positions of the bars. As A is raised higher $C\ D$ diminishes, and when the bars are parallel, we have

$$P : W :: 2\ A\ C : 0;$$

that is to say, the power has no efficiency. But as A approaches the base $A\,C$ diminishes, and at last we have, when $B\,A$ and $A\,D$ are in the same line,

$$P : W :: 0 : B\,A.$$

Hence the weight or resistance in such case is infinite as compared with the power applied. The indefinite increase of efficiency in the power, which occurs during a single movement, renders this machine one of the most useful for many purposes, as printing and coining.

Questions on the knee-joint.—

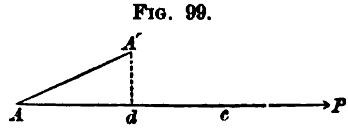
1. A power of 50 lbs. is exerted on the joint A (Fig. 97); compare the weight which will balance it, when $B\,A\,D$ is 90°. and when it is 160°. *Ans.* 25 lbs. and 141.78 lbs.

2. When the angle between the bars is 110°, a certain power just overcomes a weight of 65 lbs.; what must be the angle, in order that the weight overcome may be five times as great? *Ans.* 164° 3′ 22″.

PRINCIPLE OF VIRTUAL VELOCITIES.

140. Definition.—The *virtual velocity* of a point, with respect to any force, is the product of its actual velocity by the cosine of the angle which its actual path makes with the direction of the force. Thus, let a point A (Fig. 99) be acted upon by a force P in the direction $A\,c$, and because of some other external force or resistance suppose the point to be constrained to move in the line $A\,A'$ to A' in any unit of time: then $A\,d$, the projection of $A\,A'$ upon $A\,c$, is the virtual velocity of the point A with reference to the force P.

FIG. 99.



141. The Point of Application Moving in the Line of the Force.—It can be shown, in every case, that *the velocities, when reckoned in the direction in which the forces act, are inversely as the forces.*

Some examples are first given in which the point of application moves in the line in which the force acts.

In the *straight lever* (Fig. 100), which is in equilibrium by the weights P and W, suppose a slight motion to exist; then the velocity of each will be as the arc described in the same time; but the arcs are similar, since they subtend

FIG. 100.

equal angles. Therefore, if V = velocity of P, and v = velocity of W.

$$V : v :: A P : B W :: A C : B C;$$

but it has been shown (Art. 106) that

$$P : W :: B C : A C;$$
$$\therefore V : v :: W : P;$$

that is, the velocity of the power is to the velocity of the weight as the weight to the power. Hence, $P \times$ its velocity $= W \times$ its velocity; that is, the momentum of the power equals the momentum of the weight.

In the *wheel and axle*, let R and r be the radii, and suppose the machine to be revolved; then while P descends a distance equal to the circumference of the wheel $= 2 \pi R$, the weight ascends a distance equal to the circumference of the axle $= 2 \pi r$. Therefore,

$$V : v :: 2 \pi R : 2 \pi r :: R : r;$$

but (Art. 113), $P : W :: r : R$;

$$\therefore V : v :: W : P;$$

or, the velocities are inversely as the weights; and $P \times V = W \times v$, the momentum of the power equals the momentum of the weight.

In the *fixed pulley* the velocities are obviously equal; and we have before seen that the power and weight are equal; therefore the proportion holds true, $V : v :: W : P$; and the momenta are equal.

In the *movable pulley*, if n is the number of sustaining parts of the cord, when W rises any distance $= x$, each portion of cord is shortened by the distance x, and all these n portions pass over to P, which therefore descends a distance $= n x$.

Hence,

$$V : v :: n x : x :: n : 1;$$

but (Art. 120), $P : W :: 1 : n$;

$$\therefore V : v :: W : P;$$

as in all the preceding cases.

In the screw (Fig. 94), while the power describes the circumference $= 2 \pi \times B C$, the weight moves only the distance $= d$; therefore,

$$V : v :: 2 \pi \times B C : d;$$

but (Art. 131), $P : W :: d : 2 \pi \times B C$;

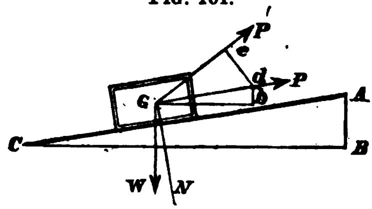
$$\therefore V : v :: W : P;$$

therefore the momentum of the power equals the momentum of the weight, as before.

142. The Point of Application Moving in a Different Line from that in which the Force Acts.—The cases thus far noticed are the most obvious ones, because the points of application of force and weight *actually move* in the directions in which their force is exerted. The case of the inclined plane will illustrate the principle, when the point of application does not move in the direction of the force.

First, let P (Fig. 101) act parallel to the plane, and suppose the body to be moved either up or down the plane a distance equal to $G\,d$. That is the velocity of the *force*. But in the direction of the *weight* (force of gravity) the body moves only the distance $b\,d$. Therefore the velocity of the force is to the velocity of the weight (each being reckoned in the line of its action) as $G\,d$ to $b\,d$.

FIG. 101.



$$\begin{array}{llcl} \text{By similar triangles,} & G\,d : b\,d & :: & A\,C : A\,B; \\ & \text{or} \quad V : v & :: & A\,C : A\,B. \\ \text{But (Art. 125),} & P : W & :: & A\,B : A\,C; \\ & \therefore V : v & :: & W : P. \end{array}$$

Again, let the force act in any oblique direction, as $G\,e$. If the body moves over $G\,d$, draw $d\,e$ perpendicular to $G\,e$; then $G\,e$ is the distance passed over in the direction of the force, and $b\,d$ in the direction of the weight. Assuming the unit of time to be consumed in the motion, these lines represent the velocities and we have

$$V : v = \frac{G\,e}{G\,d} : \frac{b\,d}{G\,d} = (\cos e\,G\,d =) \sin P\,G\,N : \sin a.$$

But (Art. 126), $\quad P : W = \sin a : \sin P\,G\,N;$

$$\therefore V : v = W : P.$$

Consider in the foregoing that the machines act for a unit time. Then V and v represent the distances through which the force and resistance have moved. $P\,V$ represents the work done by the force; $W\,v$ the work performed upon the resisting body. These are always equal, and hence a machine does not give out more work than is impressed upon it. However, the resisting force may be of any desired magnitude, and the machine will enable a smaller force to produce a given motion against it, by itself moving a greater distance. On the other hand, by a machine, a large force with small velocity can be made to produce *great velocity* against a small resistance.

FRICTION IN MACHINERY.

143. The Power and Weight not the only Forces in a Machine.—For each machine a certain proportion has been given, which insures equilibrium. And it is implied that if either the power or the weight be altered, the equilibrium will be destroyed. But practically this is not true; the power or weight may be considerably changed, or possibly one of them may be entirely removed, and the machine still remain at rest. The obstruction which prevents motion in such cases, and which always exists in a greater or less degree, arises from *friction;* and friction is caused by roughness in the surfaces which rub against each other. The minute elevations of one surface fall in between those of the other, and directly interfere with the motion of either, while they remain in contact. Polishing diminishes the friction, but can never remove it, for it never removes all roughness.

The *coefficient* of friction *is the fraction whose numerator is the force required to overcome the friction, and its denominator the normal pressure between the bodies.*

Let μ = coefficient; P_n = normal pressure; F = force required to overcome friction. Then

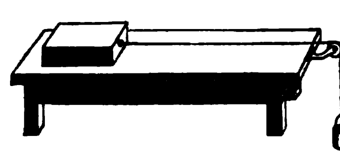
$$F = \mu P_n. \quad \therefore \mu = F/P_n.$$

As friction always tends to prevent motion, and never to produce it, it is called a *passive* force. It assists the power, when the weight is to be kept at rest, but opposes it, when the weight is to be moved. There are other passive forces to be considered in the study of science, but no other has so much influence in the operations of machinery as friction.

144. Modes of Experimenting.—When one surface slides on another, the friction which exists is called the *sliding* friction; but when a wheel rolls along a surface, the friction is called *rolling* friction. The sliding friction occurs much more in machines than the rolling friction.

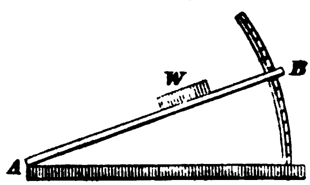
Experiments for ascertaining the laws of friction may be performed by placing on a table a block of three different dimensions, and measuring its friction under different circumstances by weights acting on the block by means of a cord and pulley, as represented in Fig. 102. This was the method by which Coulomb first ascertained the laws of friction.

FIG. 102.



Another mode is to place the block on an inclined plane, whose angle can be varied, and then find the relative friction in different cases, by the largest inclination at which it will prevent the block from sliding. For, when W on the inclined plane A B (Fig. 103) is on the point of sliding down, friction is the power which, acting parallel to the plane, is in equilibrium with that component of the weight which tends to move the block down the plane.

FIG. 103.



This component parallel to the plane, is $W \sin A$, and the normal pressure $W \cos A$; hence, calling the coefficient of friction u, we have (Art. 143) $u = \frac{W \sin A}{W \cos A} = \tan A$, or *the coefficient of friction is equal to the tangent of the angle of inclination of the plane.*

For example, suppose a block of cast iron to rest upon an oak plank, and that the end of the plank is raised so that the block slides with uniform motion down the plane; then the angle A will be found by actual measurement to be about 26°, the natural tangent of which is .48773; hence, in pounds, 49 per cent. of the weight will represent the force in pounds required to overcome the friction.

145. Laws of Sliding Friction.—The laws of sliding friction on which experimenters are generally agreed are the following:

1. *Friction varies as the pressure.*—If weights are put upon the block, it is found that a double weight requires a double force to move it, a triple weight a triple force, &c.

2. *It is the same, however great or small the surface on which the body rests.*—If the block be drawn, first on its broadest side, then on the others in succession, the force required to overcome friction is found in each case to be the same. Extremes of size are, however, to be excepted. If the loaded block were to rest on three or four very small surfaces, the obstruction might be greatly increased by the indentations thus occasioned in the surface beneath them.

3. *Friction is a uniformly retarding force.*—That is, it destroys equal amounts of motion in equal times, whatever may be the velocity, like gravity on an ascending body.

4. *Friction at the first moment of contact is less than after contact has continued for a time.*—And the time during which friction increases, varies in different materials. The friction of wood on wood reaches its maximum in three or four minutes; of metal

on metal, in a second or two; of metal on wood, it increases for several days. As any jar or vibration changes at once the friction of rest to that of motion, the coefficients to be considered in determining the stability of any structure should be those of motion.

5. *Friction is less between substances of different kinds than between those of the same kind.*—Hence, in watches, steel pivots are made to revolve in sockets of brass or of jewels, rather than of steel.

146. Friction of Axes.—In machinery, the most common case of friction is that of an axis revolving in a hollow cylinder, or the reverse, a hollow cylinder revolving on an axis. These are cases of sliding friction, in which the force that overcomes the friction, usually acts at the circumference of a wheel, and therefore at a mechanical advantage. Thus, the friction on an axis, whose coefficient is as high as 20 per cent., requires a force of only *two* per cent. to overcome it, provided the force acts at the circumference of a wheel whose diameter is ten times that of the axis.

147. Rolling Friction.—This form of friction is very much less than the sliding, since the projecting points of the surfaces do not directly encounter each other, but those of the rolling wheel are lifted up from among those of the other surface, as the wheel advances.

By the use of the apparatus described in Art. 144, the laws of the rolling are found to be the same as those of the sliding friction. But on account of the manner in which this form of friction is overcome, there is this additional law:

The force required to roll the wheel varies inversely as the diameter.

For the force, acting at the centre of the wheel to turn it on its lowest point as a momentary fulcrum, has the advantage of greater acting distance as the diameter increases.

It is the rolling friction which gives value to *friction wheels*, as they are called. When it is desirable that a wheel should revolve with the least possible friction, each end of its axis is made to rest in the angle between two other wheels placed side by side, as shown in Fig. 104. The wheel is obstructed only by the rolling friction on the surfaces of the four wheels, and the retarding effect of the sliding friction at the pivots of the latter is greatly reduced on the principle of the wheel and axle.

The sliding friction is diminished by lubricating the surface, the rolling friction is not.

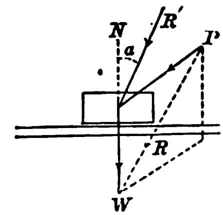
Fig. 104.

148. Advantages of Friction.—Friction in machinery is generally regarded as an evil, since more force is on this account required to do the work for which the machine is made. But it is easy to see that, in general, friction is of incalculable value, or rather, that nothing could be accomplished without it. Objects stand firmly in their places by friction ; and the heavier they are, the more firmly they stand, because friction increases with the pressure. All fastening by nails, bolts, and screws, is due to friction. The fibres of cotton, wool, or silk, when intertwined with each other, form strong threads or cords, only because of the power of friction. Without friction it would be impossible to walk, or even to stand, or to hold anything by grasping it with the hand.

149. Limiting Angle of Friction.—If two surfaces are in contact and forces be applied, oblique to the surface of contact, no motion will result so long as the resultant of all forces makes with the normal an angle whose tangent is less than the coefficient of friction, no matter how great this resultant force may be.

Thus (Fig. 105) let a block of iron rest upon a surface of oak, as in the case heretofore considered, and let the force P be applied. In this case the forces are P and W, and their resultant is R (or R') which may be considered in their stead. The component of R', which tends to produce motion, is R' sine a, and the total normal pressure is $R' \cos a$. If $\frac{R' \text{ sine } a}{R' \cos a} = \tan a$, is less than the coefficient of friction no motion can result ; that is to say, if a is less

Fig. 105.



than the inclination of the plane, in Art. 144, there will be stable equilibrium.

The greatest angle a which the resultant of all the forces can make with the normal without producing sliding motion of the surfaces is called the *limiting angle of friction*, and its tangent is equal to the coefficient of friction.

Problems on Friction.

1. A force of 6 kilos is just sufficient to move a body weighing 48 kilos uniformly along a horizontal plane: What is the coefficient of friction? *Ans.* $\frac{1}{8}$.

2. The value of μ is .3, the weight of the body is 16 kilos: What force is required to move it uniformly? *Ans.* 4.8 kilos.

3. It is found that a force of 40 grams suffices to move a body uniformly on a horizontal surface, where the value of the coefficient of friction is known to be .25: What is the weight of the body? *Ans.* 160 grams.

4. A body weighing 15 kilos is just on the point of sliding when the surface it rests upon is inclined 20°: (a) What is the coefficient of friction and the force of friction? (b) If the weight of the body is doubled, what values have these quantities?

$$Ans. \begin{cases} (a)\ \mu = .36,\ F = 5.07 \text{ kilos.} \\ (b)\ \mu = .36,\ F = 10.14 \text{ kilos.} \end{cases}$$

CHAPTER VII.

MOTION ON INCLINED PLANES.—THE PENDULUM.

150. The Force which Moves a Body Down an Inclined Plane.—It was shown (Art. 125) that when the force acts in a line parallel to the inclined plane, $P : W :: A\ B : A\ C$. If, therefore, P ceases to act, the body descends the plane only with a force equal to P.

Let g (the velocity acquired in a second in falling freely) = the force of gravity, f = the force acting down the plane, h = the height, l = the length; then by substitution,

$$f : g :: h : l, \text{ and}$$

$$f = \frac{h}{l}\, g.$$

Therefore, the force which moves a body down an inclined plane is equal to that fraction of gravity which is expressed by the height divided by the length. This is evidently a constant force on any given plane, and produces uniformly accelerated motion. Therefore the motion on an inclined plane does not differ from that of free fall in kind, but only in degree. Hence the formulæ for time, space, and velocity on an inclined plane are like those relating to free fall, if the value of f be substituted for g.

151. Formulæ for the Inclined Plane.—The formulæ for free fall (Art. 27) are here repeated, and against them the corresponding formulæ for descent on an inclined plane.

	Free fall.		*Descent on an inclined plane.*
1.	$s = \frac{1}{2} g t^2$	. . . .	$s = \frac{g h t^2}{2 l}$.
2.	$t = \sqrt{\frac{2 s}{g}}$	. . . .	$t = \sqrt{\frac{2 l s}{g h}}$.
3.	$s = \frac{v^2}{2 g}$	. . . .	$s = \frac{l v^2}{2 g h}$.
4.	$v = \sqrt{2 g s}$	. . . .	$v = \sqrt{\frac{2 g h s}{l}}$.
5.	$t = \frac{v}{g}$.	. . . .	$t = \frac{l v}{g h}$.
6.	$v = g t$	. . . .	$v = \frac{g h t}{l}$.

By formula 1, $s \propto t^2$, and by formula 3, $s \propto v^2$. It follows that in equal successive times the spaces of descent are as the odd numbers, 1, 3, 5, &c., and of ascent as these numbers inverted; also, that with the acquired velocity continued uniformly, a body moves twice as far as it must descend to acquire that velocity. If a body be projected up an inclined plane, it will ascend as far as it must descend in order to acquire the velocity of projection. The distance passed over in the time t by a body projected with the velocity v, down or up an inclined plane, equals $t\, v \pm \frac{g h t^2}{2 l}$.

152. Formulæ for the whole Length of a Plane.—

1. *The velocity acquired in descending a plane is the same as that acquired in falling down its height.*

For now $s = l$; hence (formula 4), $v = \left(\frac{2 g h s}{l}\right)^{\frac{1}{2}} = (2 g h)^{\frac{1}{2}}$, which is the formula for free fall through h, the height of the plane.

On different planes, therefore, $v \propto h^{\frac{1}{2}}$.

2. *The time of descending a plane is to the time of falling down its height as the length to the height.*

For (formula 2) $t = \left(\frac{2\,l\,s}{g\,h}\right)^{\frac{1}{2}} = l\left(\frac{2}{g\,h}\right)^{\frac{1}{2}}$. But the time of fall down the height is $\left(\frac{2\,h}{g}\right)^{\frac{1}{2}}$. Therefore,

$$t \text{ down plane} : t \text{ down height} :: l\left(\frac{2}{g\,h}\right)^{\frac{1}{2}} : \left(\frac{2\,h}{g}\right)^{\frac{1}{2}};$$

$$:: l\left(\frac{2}{g}\right)^{\frac{1}{2}} : h\left(\frac{2}{g}\right)^{\frac{1}{2}};$$

$$:: l : h.$$

On different planes, $t \propto \frac{l}{\sqrt{h}}$.

It follows that if several planes have the same height, the velocities acquired in descending them are equal, and the times of descent are as the lengths of the planes. For, let $A\,C$, $A\,D$, $A\,E$, (Fig. 106) have the same height $A\,B$; then, since $v \propto h^{\frac{1}{2}}$, and h is the same for all, v is the same. And since $t \propto \frac{l}{\sqrt{h}}$, and h is the same for all the planes, $t \propto l$.

FIG. 106.

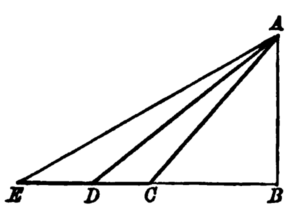
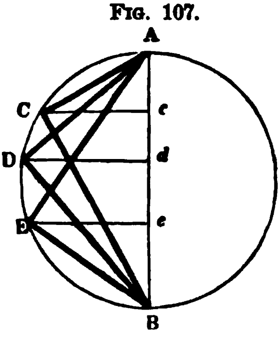


FIG. 107.



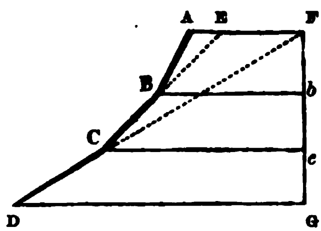
153. Descent on the Chords of a Circle.—In descending the chords of a circle which terminate at the ends of the vertical diameter, *the acquired velocities are as the lengths, and the times of descent are equal to each other and to the time of falling through the diameter.*

For (Art. 152) the velocity acquired on $A\,C$ (Fig. 107) $= (2\,g \cdot A\,c)^{\frac{1}{2}} = \left(2\,g \cdot \frac{A\,C^2}{A\,B}\right)^{\frac{1}{2}} = A\,C\left(\frac{2\,g}{A\,B}\right)^{\frac{1}{2}}$, which, since $\left(\frac{2\,g}{A\,B}\right)^{\frac{1}{2}}$ is constant, varies as $A\,C$, the length.

Again (Art. 152), the time down $A\ C = \left(\frac{2\ A\ C}{g\ .\ A\ c}\right)^{\frac{1}{2}} = \left(\frac{2\ A\ B\ .\ A\ c}{g\ .\ A\ c}\right)^{\frac{1}{2}} = \left(\frac{2\ A\ B}{g}\right)^{\frac{1}{2}}$, which is equal to the time of falling freely through $A\ B$, the diameter.

154. Velocity Acquired on a Series of Planes.—If no velocity be lost in passing from one plane to another, the velocity acquired in descending a series of planes is equal to that acquired in falling through their perpendicular height. For, in Fig. 108, the velocity at B is the same, whether the body comes down $A\ B$ or $E\ B$, as they are of the same height, $F\ b$. If, therefore, the body enters on $B\ C$ with the acquired velocity, then it is immaterial whether the descent is on $A\ B$ and $B\ C$ or on $E\ C$; in either case, the velocity at C is equal to that acquired in falling $F c$. In like manner, if the body can change from $B\ C$ to $C\ D$ without loss of velocity, then the velocity at D is the same, whether acquired on $A\ B$, $B\ C$, and $C\ D$, or on $F\ D$, which is the same as down $F\ G$.

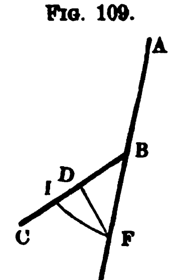
FIG. 108.



155. The Loss in Passing from one Plane to Another.—The condition named in the foregoing article is not fulfilled. A body *does* lose velocity in passing from one plane to another. And the *loss* is to the *whole previous velocity* as the *versed sine of the angle* between the planes to *radius*.

Let $B\ F$ (Fig. 109) represent the velocity which the body has at B. Resolve it into $B\ D$ on the second plane, and $D\ F$ perpendicular to it. $B\ D$ is the initial velocity on $B\ C$; and, if $B\ I = B\ F$, $D\ I$ is the loss. But $D\ I$ is the versed sine of the angle $F\ B\ D$, to the radius $B\ F$; and $\therefore$ the loss is to the velocity at B as $D\ I : B\ F ::$ ver. sin B : rad.

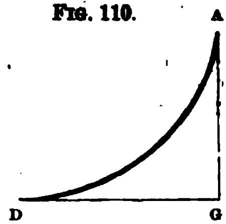
FIG. 109.



156. No Loss on a Curve.—Suppose now the number of planes in a system to be infinite; then it becomes a curve (Fig. 110). As the angle between two successive elements of the curve is infinitely small, its chord is also infinitely small; but its versed sine is *infinitely smaller still*, i. e., an infinitesimal of the *second* order; for diam. : chord :: chord : ver. sin. Therefore, although the sum of all the infinitely small angles

is a finite angle $180° - A\ G\ D$, yet, as the loss of velocity at each point is an infinitesimal of the *second* order, the *entire loss* (which is the sum of the losses at all points of the curve) is an infinitesimal of the *first* order.

FIG. 110.



Hence, a body loses no velocity on a curve, and therefore acquires at the bottom the same velocity as in falling freely through its height.

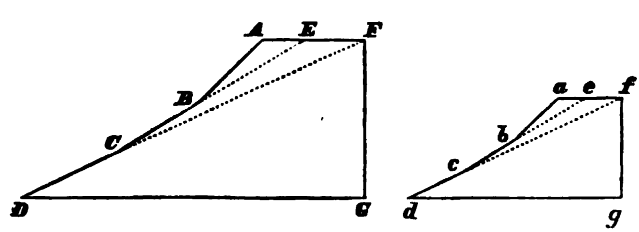
It appears, therefore, that whether a body descends *vertically*, or on an *inclined plane*, or on a *curve* of any kind, the *acquired velocity is the same*, if the height is the same.

157. Times of Descending Similar Systems of Planes and Similar Curves.—If planes are equally inclined to the horizon, the *times* of describing them are as *the square roots of their lengths*. For, if the height and base of each plane be drawn, similar triangles are formed, and $h : l$ is a constant ratio for the several planes. By Art. 152, $t \propto \frac{l}{\sqrt{h}} \propto \frac{l}{\sqrt{l}} \propto \sqrt{l}$; that is, the time varies as the square root of the length.

If two *systems* of planes are similar, i. e., if the corresponding parts are proportional and equally inclined to the horizon, it is still true that *the times of descending them are as the square roots of their lengths.*

Let $A\ B\ C\ D$ and $a\ b\ c\ d$ (Fig. 111) be similar, and let $A\ F$ and

FIG. 111.



$a\ f$ be drawn horizontally, and the lower planes produced to meet them, then it is readily proved that all the homologous lines of the figures are proportional, and their square roots also proportional. Then (reading $t, A\ B$, *time down* $A\ B$, &c.),

we have $$t, A\ B : t, a\ b :: \sqrt{A\ B} : \sqrt{a\ b};$$

$$t, E\ B : t, e\ b :: \sqrt{E\ B} : \sqrt{e\ b} :: \sqrt{A\ B} : \sqrt{a\ b};$$

and $\quad t, EC : t, ec :: \sqrt{EC} : \sqrt{ec} :: \sqrt{AB} : \sqrt{ab};$

$\therefore$ (by subtraction) $t, BC : t, bc :: \sqrt{AB} : \sqrt{ab}.$

In like manner, $t, CD : t, cd :: \sqrt{AB} : \sqrt{ab}.$

$\therefore$ (by addition)

$$t, (AB + BC + CD) : t, (ab + bc + cd) :: \sqrt{AB} : \sqrt{ab}$$
$$:: \sqrt{(AB + BC + CD)} : \sqrt{(ab + bc + cd)}.$$

Though there is a loss of velocity in passing from one plane to another, the proposition is still true; because, the angles being equal, the losses are proportional to the acquired velocities; and therefore the initial velocities on the next planes are still in the same ratio as before the losses; hence the ratio of times is not changed.

The reasoning is applicable when the number of planes in each system is infinitely increased, so that they become *curves*, similar, and similarly inclined to the horizon. Suppose these curves to be *circular* arcs; then, as they are similar, they are proportional to their radii. Hence, the times of descending similar circular arcs are as the square roots of the radii of those arcs.

158. Questions on the Motions of Bodies on Inclined Planes.—

1. How long will it take a body to descend 100 feet on a plane whose length is 150 feet, and whose height is 60 feet?

Ans. 3.9 sec.

2. There is an inclined railroad track, $2\frac{1}{4}$ miles long, whose inclination is 1 in 35. What velocity will a car acquire, in running the whole length of the road by its own weight?

Ans. 106.2 miles per hour.

3. A body weighing 5 lbs. descends vertically, and draws a weight of 6 lbs. up a plane whose inclination is 45°. How far will the first body descend in 10 seconds? *Ans.* 110.74 feet.

159. The Pendulum.—A pendulum is a weight attached by an inflexible rod to a horizontal axis of suspension, so as to be free to vibrate by the force of gravity. If it is drawn aside from its position of rest, it descends, and by the momentum acquired, rises on the opposite side to the same height, when gravity again causes its descent as before. If unobstructed, its vibrations would never cease.

A *single vibration* is the motion from the highest point on one side to the highest point on the other side. The motion from the highest point on one side to the same point again is called a *double vibration.*

The *axis of the pendulum* is a line drawn through its centre of gravity perpendicular to the horizontal axis about which the pendulum vibrates.

The *centre of oscillation* of a pendulum is that point of its axis at which, if the entire mass were collected, its time of vibration would be unchanged.

The *length* of a pendulum is that part of its axis which is included between the axis of suspension and the centre of oscillation.

All the particles of a pendulum may be conceived to be collected in points lying in the axis. Those which are *above* the centre of oscillation tend to vibrate quicker (Art. 157), and therefore accelerate it; those which are *below* tend to vibrate slower, and therefore retard it. But, according to the definition of the centre of oscillation, these accelerations and retardations exactly balance each other at that point.

160. Calculation of the Length of a Pendulum.—Let $C\,q$ (Fig. 112) be the axis of a pendulum in which all its weight is collected, C the point of suspension, G the centre of gravity, O the centre of oscillation, a b, &c., particles above O, which accelerate it, p, q, &c., particles below O, which retard it. $C\,O = l$, is the length of the pendulum required. Denote the masses concentrated in $a, b \ldots . p, q$, by $m, m' \ldots . m'', m'''$, and their distances from C by $r, r' \ldots , . r''$, r'''; and denote the distance from C to G by k. Denote the angular velocity, that is, the velocity at unit's distance from the centre, at any instant by θ; then the velocity of m will be $r\,\theta$ and its momentum will be $m\,r\,\theta$.

FIG. 112.

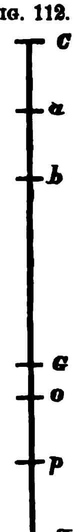
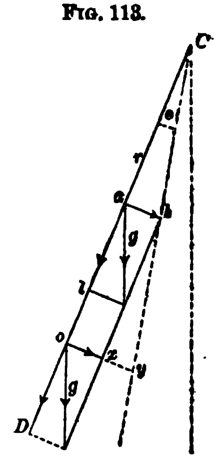


FIG. 113.



If m had been placed at O, the momentum would have been $m\,l\,\theta$. The difference $m\,(l - r)\,\theta$, is that portion of the force which accelerates the motion of the system.

For suppose a material particle m (Fig. 113) to act upon a pendulum $C\,D$ without weight; m at a would, under the action of the component of gravity $a\,b$, move the *point* a to b and swing the pendulum through the angle θ; m if transferred to o would,

gravity being the same, move the *point* o to x, and swing the pendulum through an angle less than θ. Thus m at a swings the pendulum through a greater angle in a given time than it would if at o, or accelerates the pendulum, by a force which would carry m over $x\,y$ in the given time, or by $m\,(l - r)\,\theta$; for, calling $C\,a = r$ and $C\,o = l$, $a\,b = r\,\theta$, $o\,x = a\,b = r\,\theta$, $o\,y = l\,\theta$; then $o\,y - o\,x = x\,y = l\,\theta - r\,\theta = (l - r)\,\theta$, and m moving with velocity $x\,y$, or $(l - r)\,\theta$, gives momentum $m\,(l - r)\,\theta$.

The moment of this force with respect to C is $m\,(l - r)\,r\,\theta$.

In like manner the moment of m' is $m'\,(l - r')\,r'\,\theta$, and so on for all the particles between C and O.

The moments of the forces tending to retard the system applied at the points p, q, &c., are

$$m''\,(r'' - l)\,r''\,\theta,\ m'''\,(r''' - l)\,r'''\,\theta,\ \&c.$$

But since these forces are to balance each other, we have

$$m\,(l - r)\,r\,\theta + m'\,(l - r')\,r'\,\theta + \&c. = m''\,(r'' - l)\,r''\,\theta + m'''\,(r''' - l)\,r'''\,\theta + \&c.;$$

whence $$l = \frac{m\,r^2 + m'\,r'^2 + m''\,r''^2 + \&c.}{m\,r + m'\,r' + m''\,r'' + \&c.}$$

Or $l = \dfrac{S\,(m\,r^2)}{S\,(m\,r)}$, where S denotes the sum of all the terms similar to that which follows it.

The numerator of this expression is called the *moment of inertia* of the body with respect to the axis of suspension, and the denominator is called the *moment of the mass*, with respect to the axis of suspension.

By the principle of moments $m\,r + m'\,r' + \&c.$, or $S\,(m\,r) = M\,k$, where M denotes the entire mass of the pendulum; hence, by substitution, $l = \dfrac{S\,(m\,r^2)}{M\,k}$.

That is, *the distance from the axis of suspension to the centre of oscillation is found by dividing the moment of inertia, with respect to that axis, by the moment of the mass with respect to the same axis.*

161. The Point of Suspension and the Centre of Oscillation Interchangeable.—Let the pendulum now be suspended from an axis passing through O, and denote by l' the distance from O to the new centre of oscillation. The distances of $a, b \ldots p, q$, from O, will be $l - r$, $l - r'$, &c., and the distance $G\,O$ will be $l - k$.

Hence, from the principle just established, we have

$$l' = \frac{S\,[m\,(l - r)^2]}{M\,(l - k)} = \frac{S\,(m\,l^2 - 2\,m\,r\,l + m\,r^2)}{M\,(l - k)}$$

$$= \frac{S\,(m\,l^2) - 2\,S\,(m\,r\,l) + S\,(m\,r^2)}{M\,(l - k)}.$$

But from the preceding paragraph $l = \dfrac{S\,(m\,r^2)}{M\,k}$, whence, $S\,(m\,r^2) = M\,k\,l$; and since l is constant, $S\,(m\,l^2) = (m + m' + m'' + \&c.)\,l^2 = M\,l^2$, which values substituted above give

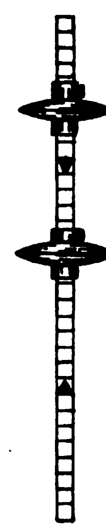
$$l' = \frac{M\,l^2 - 2\,l\,S\,(m\,r) + M\,k\,l}{M\,(l - k)} = \frac{M\,l^2 - 2\,M\,k\,l + M\,k\,l}{M\,(l - k)}$$

$$= \frac{M\,(l - k)\,l}{M\,(l - k)} = l.$$

This last equation shows that the centre of oscillation and the point of suspension are interchangeable; that is, if the pendulum were suspended from O, it would vibrate in the same time as when suspended from C.

This fact is taken advantage of in determining the length of the second's pendulum at any place. A solid bar (Fig. 114) is furnished with two knife-edge axes A and B, and two sliding weights C and D. By adjusting these weights the bar can be made to oscillate in the same time when suspended upon either axis. The distance between the knife-edges A and B is the length of an equivalent simple pendulum, and by comparing the time of oscillation with that of a pendulum beating seconds, the time of one oscillation of this reversible pendulum is obtained; from these data the length of the second's pendulum is readily computed.

FIG. 114.



162. Calculation of the Time of Oscillation.—Let the length of the pendulum $A\,B$ (Fig. 115) be represented by l, and the height of the arc of oscillation by $B\,D$. Suppose the pendulum to have moved from C to a; its acquired velocity will be $v = \sqrt{2\,g \times D\,H}$. (Art. 156.)

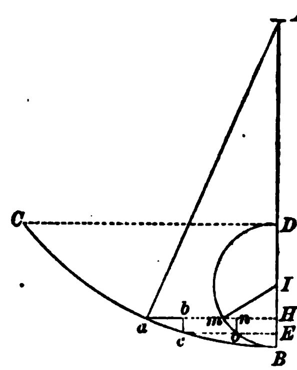
During the succeeding infinitely small interval of time t' it will describe the element of its arc $a\,c$ with the velocity v; hence

$$t' = \frac{a\,c}{v} = \frac{a\,c}{\sqrt{2\,g \times D\,H}}.$$

Describe upon $D\,B$ a semi-circumference; $m\,o$ is the elemen-

tary arc of the semi-circumference having the same altitude $E\,H$ as the element $a\,c$. As these arcs are elements of the curves, they may be regarded as straight lines, and $a\,b\,c$ and $m\,n\,o$ become triangles. $A\,H\,a$ and $a\,b\,c$ are similar triangles, their corresponding sides being perpendicular two and two, and give the equation $\frac{a\,c}{c\,b} = \frac{A\,a}{a\,H}$, and because the triangles $m\,n\,o$ and $I\,m\,H$ are similar, for the reason assigned above, $\frac{m\,o}{n\,o} = \frac{I\,m}{m\,H}$.

FIG. 115.



Divide the first of these equations by the second, and we get $\frac{a\,c}{m\,o} = \frac{A\,a \times m\,H}{a\,H \times I\,m}$.

But $\overline{a\,H}^2 = B\,H\,(2\,A\,B - B\,H) = B\,H\,(2\,l - B\,H)$ and $\overline{m\,H}^2 = B\,H \times H\,D$, whence,

$$\frac{a\,c}{m\,o} = \frac{l\sqrt{B\,H \times H\,D}}{I\,m\sqrt{B\,H\,(2\,l - B\,H)}} = \frac{l\sqrt{H\,D}}{I\,m\sqrt{2\,l - B\,H}}, \text{ or since}$$

$$I\,m = \tfrac{1}{2}\,B\,D \text{ and } B\,H = B\,D - D\,H,$$

$$a\,c = \frac{l\sqrt{H\,D}}{\frac{1}{2}\,B\,D\sqrt{2\,l - (B\,D - D\,H)}} \times m\,o,$$

and this substituted in the value of t', gives

$$t' = \frac{l\sqrt{H\,D}}{\frac{1}{2}\,B\,D\sqrt{2\,l - (B\,D - D\,H)}} \times \frac{m\,o}{\sqrt{2\,g \times D\,H}}$$

$$= \frac{2\,l \times m\,o}{B\,D\sqrt{2\,g\,[2\,l - (B\,D - D\,H)]}};$$

or dividing both terms by $2\sqrt{l}$, we obtain

$$t' = \frac{\sqrt{l} \times m\,o}{B\,D \times \sqrt{g}\sqrt{1 - \frac{B\,D - D\,H}{2\,l}}}.$$

When the amplitude is small, the double arc $B\,C$ not being over 5 degrees, $\frac{B\,D}{2\,l}$, and $\frac{D\,H}{2\,l}$ are very small, and their difference, $\frac{B\,D - D\,H}{2\,l}$, is smaller still, and may be neglected, giving as a

result $t' = \sqrt{\frac{l}{g}} \times \frac{m\,o}{B\,D}$. The time required to describe $C\,B$ is the sum of the times of describing the elements of $C\,B$, or calling this time $\frac{t}{2}$, we have $\frac{t}{2} = \sqrt{\frac{l}{g}} \times \frac{1}{B\,D} \times$ (sum of the elementary arcs $m\,o$).

But the sum of the elements of $D\,m\,B$ corresponding to the elements of $C\,B$ is the semicircle $D\,m\,B$ itself, or $\pi\,\frac{B\,D}{2}$; whence $\frac{t}{2} = \frac{\pi}{2}\sqrt{\frac{l}{g}}$ = time of semi-oscillation, or calling t the time of a complete oscillation, we have

$$t = \pi\sqrt{\frac{l}{g}}.$$

163. Applications of the Formula.—From the equation $t = \pi\sqrt{\frac{l}{g}}$, we get $l = \frac{g\,t^2}{\pi^2}$. Therefore, the length of a pendulum being known, the time of one vibration is found; and on the other hand, if the time of a vibration is known, the length of the pendulum is obtained from it.

From the same formulæ, we find that $t \propto \sqrt{l}$, or

The time in which a pendulum makes a vibration varies as the square root of the length.

As $t \propto \sqrt{l}$, $\therefore l \propto t^2$; hence, if the length of a seconds pendulum equals l, then a pendulum which vibrates *once in two seconds* equals $4\,l$, and one which beats *half seconds* $= \frac{1}{4}\,l$, &c.

Again, by observing the *length* of a pendulum which vibrates in a given *time*, the *force of gravity*, g, may be found. For, as $l = \frac{g\,t^2}{\pi^2}$, $\therefore g = \frac{\pi^2\,l}{t^2}$. And if g varies, as it does in different latitudes and at different altitudes, then $l = \frac{g\,t^2}{\pi^2} \propto g\,t^2$; and if the time is constant (as, for example, *one second*), then $l \propto g$. Hence,

The length of a pendulum for beating seconds varies as the force of gravity.

Also, $t \propto \left(\frac{l}{g}\right)^{\frac{1}{2}}$; that is, the time of a vibration varies directly as the square root of the length, and inversely as the square root of the force of gravity.

Since the *number*, n, of vibrations in a given time varies inversely as the *time* of one vibration, therefore $n \propto \left(\frac{g}{l}\right)^{\frac{1}{2}}$, and

$g \propto l\,n^2$. Hence, if the time and the length of a pendulum are given,

The force of gravity varies as the square of the number of vibrations.

1. What is the length of a pendulum to beat seconds, at the place where a body falls $16\frac{1}{12}$ ft. in the first second?

Ans. 39.11 inches, nearly.

2. If 39.11 inches is taken as the length of the seconds pendulum, how long must a pendulum be to beat 10 times in a minute?

Ans. $117\frac{1}{3}$ feet.

3. In London, the length of a seconds pendulum is 39.1386 inches; what velocity is acquired by a body falling one second in that place?

Ans. 32.19 feet.

164. The Compensation Pendulum.—This name is given to a pendulum which is so constructed that its length does not vary by changes of temperature. As all substances expand by heat and contract by cold, therefore a pendulum will vibrate more slowly in warm than in cold weather. This difficulty is overcome in several ways, but always by employing two substances whose rates of expansion and contraction are unequal. One of the most common is the *gridiron pendulum*, represented in Fig. 116. It consists of alternate rods of steel and brass, connected by cross-pieces at top and bottom. The rate of longitudinal expansion and contraction of brass to that of steel is about as 100 to 61; so that *two* lengths of brass will increase and diminish more than *three* equal lengths of steel. Therefore, while there are three expansions of steel downward, two upward expansions of brass can be made to neutralize them. In the figure the dark rods represent steel, the white ones brass. Suppose the temperature to rise, the two outer steel rods (acting as one) let down the cross-bar d; the two brass rods standing on d raise the bar b; the steel rods suspended from b let down the bar e, on which the inner brass rods stand, and raise the short bar c; and finally, the centre steel rod, passing freely through d and e, lets down the disk of the pendulum. These lengths (counting each pair as a single rod) are adjusted so as to be in the ratio of 100 for the steel to 61 for the brass; in which case the upward expansions just equal those which are downward, and therefore

FIG. 116.

the centre of oscillation remains at the same distance from the point of suspension.

If the temperature falls, the two contractions of brass are equal to the three of steel, so that the pendulum is not shortened by cold.

The *mercurial pendulum* consists of a steel rod terminating at the bottom with a rectangular frame in which is a tall narrow jar containing mercury, which is the weight of the pendulum. It requires only 6.31 inches of mercury to neutralize the expansions and contractions of 42 inches of steel. See Appendix for calculations of the place of the centre of oscillation.

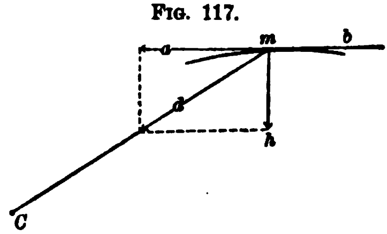
CHAPTER VIII.

CENTRAL FORCES.

165. Central Forces Described.—When two forces act upon a body—one an impulse, which alone would cause uniform motion in a straight line; the other a continued force, which urges the body toward some point not in the line of motion due to the impulse—the resultant motion will be in a curve. The first is called the *projectile* force, the second the *centripetal* force.

Suppose a point m (Fig 117) to be acted upon by an impulse, in direction and intensity represented by $b\ m$, and also by a constant force, $m\ d$. This centripetal force $m\ d$ may be resolved into two components; one $m\ a$ in the direction of the tangent, the other, $m\ h$, perpendicular to it. The *tangential* component will accelerate or retard the motion in the curved path according as it acts with the projectile force, or in opposition to it, while the component at right angles to this tends to deflect the body from a rectilinear path, and therefore determines the character of the curve at any instant.

FIG. 117.



When the body moves in the circumference of a circle, the *tangential* component of the centripetal force is 0, and hence the motion is uniform.

If the centripetal force should cease to act at any instant, the

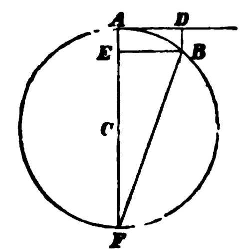
body, by its inertia, would immediately begin to move in a straight line tangent to the curve at the point where the body was when the force ceased to act.

Since the body, by its inertia, *tends* to move in a tangent, there is a continued resistance to deflection into a curved path, equal and opposed to the component $m\ h$, in the direction of the radius of curvature at the instant; this is called the *centrifugal* force, and, like gravity, is an *accelerating force.*

166. Expressions for the Centrifugal Force in Circular Motion.—

1. Let r = the radius of the circle, v = the velocity of the body, f = the acceleration of the centrifugal force, and let $A\ B$ (Fig. 118) be the arc described in the infinitely small time t; then $A\ B = v\ t$, and by a method similar to that employed in the discussion of the force of gravity, it may be shown that $A\ E = B\ D = \frac{1}{2} f\ t^2$.

FIG. 118.



But $A\ B$, being a very small arc, may be considered as equal to its chord, which is a mean proportional between $A\ E$ and the diameter $2\ r$. Hence

$$A\ E = \frac{A\ B^2}{2\ r}, \text{ or } \tfrac{1}{2} f\ t^2 = \frac{v^2\ t^2}{2\ r}.$$

Reducing we have the acceleration of the centrifugal force

$$f = \frac{v^2}{r}. \quad \ldots \ldots \ldots \ldots \quad (1)$$

From (1) it follows, that *in equal circles the acceleration varies as the square of the velocity.*

2. The value of f may be expressed in a different form. Let t' = the time of a complete revolution, then $2\ \pi\ r = v\ t'$; whence $v = \frac{2\ \pi\ r}{t'}$. This substituted in (1) gives

$$f = \frac{4\ \pi^2\ r}{t'^2}. \quad \ldots \ldots \ldots \ldots \quad (2)$$

Hence, *The acceleration varies directly as the radius of the circle, and inversely as the square of the time of revolution.*

3. If the acceleration f is imparted to a mass m, the centrifugal force exerted is (Art. 12),

$$F = m\ f = \frac{m\ v^2}{r}.$$

Introducing the weight $w = m\ g$, we have

$$F = \frac{w}{g} \frac{v^2}{r}. \quad \ldots \ldots \ldots \ldots \quad (3)$$

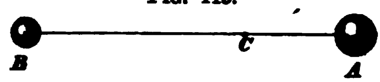
Let n = the number of revolutions per second; then

$$v = 2\pi r n, \text{ and (3) becomes}$$

$$F = \frac{4\pi^2}{g} . w . r . n^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

167. Two Bodies Revolving about their Centre of Gravity.—Let A and B (Fig. 119) be two masses connected by a rod, and let them be made to revolve about the centre of gravity C; then by (4) the centrifugal force of A will be

Fig. 119.

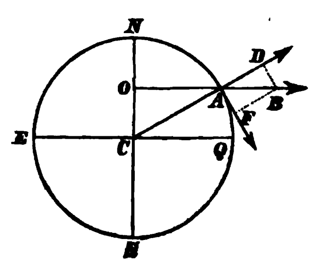


$\frac{4\pi^2}{g} . A . A\,C . n^2$, and of B, $\frac{4\pi^2}{g} . B . B\,C . n^2$.

But C being the centre of gravity of the two bodies, $A . A\,C = B . B\,C$; $\therefore$ the centrifugal force of A equals that of B. Hence, *If two bodies revolve in the same time about an axis passing through their centre of gravity, there will be no strain upon that axis.*

168. Centrifugal Force on the Earth's Surface.—As the earth revolves upon its axis all free particles upon it are influenced by the centrifugal force. Let $N\,S$ (Fig. 120) be the axis, and A a particle describing a circumference with the radius $A\,O$. Put $r = C\,Q$, $r' = A\,O$, l = the angle $A\,C\,Q$, the latitude, f = the centrifugal acceleration at the equator, f' = the acceleration at A, v = velocity of Q, and v' = velocity of A; then

Fig. 120.



$$f = \frac{v^2}{r}, \text{ and } f' = \frac{v'^2}{r'}.$$

But $v : v' :: r : r'$; whence $v' = \frac{v\,r'}{r}$. Again, from the triangle $A\,C\,O$ we have $r' = r \cos l$; hence $v' = v \cos l$, and $f' = \frac{v^2 \cos^2 l}{r \cos l} = \frac{v^2 \cos l}{r}$. Comparing the value of f' with that of f, we have

$$f' = f \cos l.$$

That is, *the centrifugal force at any point on the earth's surface is equal to the centrifugal force at the equator, multiplied by the cosine of the latitude of the place.*

A body at the equator loses by centrifugal force $\frac{1}{289}$ part of the weight which it would have if the earth did not revolve on its axis.

Let $A\,B$ represent the centrifugal force at A, and resolve it

into $A\ D$ on $C\ A$ produced, and $A\ F$, tangent to the meridian $N\ Q\ S$; then, since the angle $D\ A\ B = A\ C\ Q = l$, we have

$$A\ D = A\ B \cos l = f \cos l \,.\, \cos l = f \cos^2 l.$$

That is, *that component of the centrifugal force at any point, which opposes the force of gravity, is equal to the centrifugal force at the equator, multiplied by the square of the cosine of the latitude of the place.*

In like manner we find $A\ F = A\ B \sin l = f \cos l \sin l = \frac{f \sin 2\,l}{2}$. From this equation we see that the tangential component is 0 at the equator, increases till $l = 45°$; where it is a maximum; then goes on diminishing till $l = 90°$, when it again becomes 0.

The effect of $A\ D$ is to diminish the weight of the particle, while the effect of $A\ F$ is to urge it toward the equator.

169. Examples on Central Forces.—

1. A ball weighing 10 lbs. is whirled around in a circumference of 10 feet radius, with a velocity of 30 feet per second. What is the tension upon the cord which restrains the ball?

Ans. 28 lbs., nearly.

2. With what velocity must a body revolve in a circumference of 5 feet radius, in order that the centrifugal force may equal the weight of the body? *Ans.* $v = 12.7$ ft.

3. A ball weighing 2 lbs. is whirled round by a sling 3 feet long, making 4 revolutions per second. What is its centrifugal force? *Ans.* 117.84 lbs.

4. A weight of 5 lbs. is attached to the end of a cord 3 feet long just capable of sustaining a weight of 100 lbs. How many revolutions per second must the body make in order that the cord may be upon the point of breaking? *Ans.* $n = 2.3$, nearly.

5. A railway carriage, weighing 7 tons, moving at the rate of 30 miles per hour, describes an arc whose radius is 400 yards. What is the outward pressure upon the track? *Ans.* 701 + lbs.

170. Composition of two Rotary Motions.—

When a body is rotating on an axis, and a force is applied which alone would cause it to rotate on some other axis, the body will commence rotation on an axis lying between them, and the velocities of rotation on the three axes are such, that each may be represented by the sine of the angle between the other two.

Suppose a body is rotating on an axis $A\ B$ in the plane of $H\ K$, and that a force is applied to make it rotate on the axes $C\ D$ in the same plane $H\ K$, these two axes intersecting within the body at some point called G.

Imagine a perpendicular to the plane of the axes to be drawn through G, and let P be a particle of the body in this perpendicular. Suppose the particle P, in an infinitely small time t, to pass over $P\,a$ perpendicular to $A\,B$, by the first rotation, and over $P\,c$, perpendicular to $C\,D$, by the second. Then, since the particle will describe the diagonal $P\,e$ in the time t, this line must indicate the direction and velocity of the resultant rotation. Therefore, if $E\,F$ be drawn through G, perpendicular to the plane $G\,P\,e$, $E\,F$ is the axis on which the body revolves in consequence of the two rotations given to it. Since $P\,G$ is perpendicular to the plane $A\,G\,C$, and also to the line $E\,F$, therefore $E\,F$ is in that plane; that is, the new axis of rotation is in the plane of the other two axes. The angles $A\,G\,E$ and $E\,G\,C$, are respectively equal to the angles $a\,P\,e$ and $e\,P\,c$, the inclinations of the planes of rotation. But the lines, $P\,a$, $P\,c$, $P\,e$, represent the velocities in those directions respectively; and (Art. 44) $P\,a : P\,c : P\,e :: \sin c\,P\,e : \sin a\,P\,e : \sin a\,P\,c$; therefore $P\,a : P\,c : P\,e :: \sin C\,G\,E : \sin A\,G\,E : \sin A\,G\,C$: or, the velocities on the three axes, (namely, the axes of the component rotations, and of the resultant rotation,) are such, that each may be represented by the sine of the angle between the other two axes.

FIG. 121.

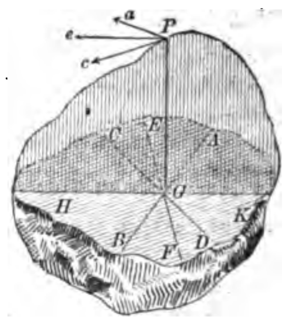
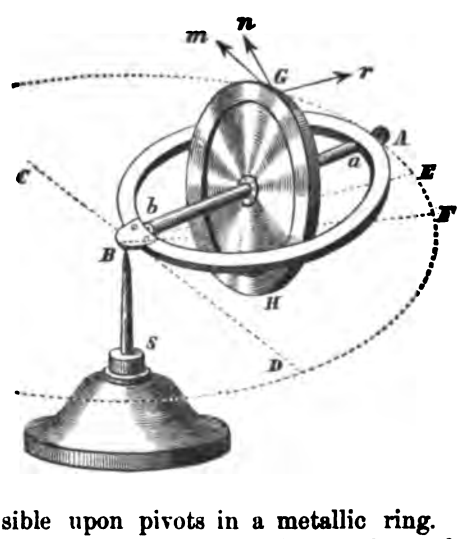


FIG. 122.



171. The Gyroscope.—The *gyroscope* affords an illustration of the composition of two rotations imparted to a body. As usually constructed, it consists of a heavy wheel $G\,H$ (Fig. 122), accurately balanced on the axis $a\,b$, which runs with as little friction as possible upon pivots in a metallic ring. In the direction of the axis, there is a projection B from the ring, having a socket

sunk into it on the under side, so that it may rest on the pointed standard *S*, without danger of slipping off.

The wheel is made to rotate swiftly by drawing off a cord wound upon *a b*, and then the socket in *B* is placed on the standard, and the whole left to itself. Immediately, instead of falling, the ring and wheel commence a slow revolution in a horizontal plane around the standard, the point *A* following the circumference *A E F*, in a direction contrary to the motion of the top of the wheel.

This revolution is explained by applying the principle of composition of rotations given in the preceding article. The particles of the wheel are rotating about the horizontal axis *a b* by the force imparted by the string. The force of gravity tends to make it fall, that is, to revolve in a vertical circle around the axis *C D* at right angles to *a b*. Hence, in a moment after dropping the ring, the system will be found revolving on an axis which lies in the direction *E B*, between *A B* and *C D*, the other two axes. *Now*, gravity bears it down around a new axis perpendicular to *E B*. Therefore, as before, it changes to still another axis *F B*, and thus continues to go round in a horizontal circle.

The only way possible for it to rotate on an axis in a new position, is to turn its present axis of rotation into that position. Hence, the whole instrument turns about, in order that its axis may take these successive positions.

The change of axis is seen also by observing the resultant of the motions of the particles at the top and bottom of the wheel. For example, *G* is moving swiftly in the direction *m* by the rotation around *a b*; by gravity it tends to move slowly in the line *r*, tangent to a vertical circle about the centre *B*. The resultant is in the line *n*, tangent to the wheel when its axis *a b* has taken the new position *E B*.

The centre of gravity of the ring and wheel tends to remain at rest, while the resultant of the two rotations carries around it all other parts, standard included, in horizontal circles. But the standard by its inertia and friction resists this effort, and the reaction causes the ring and wheel to go around the standard.

PART II.

HYDRAULICS.

CHAPTER I.

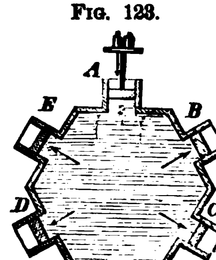
HYDROSTATICS.

172. Liquids Distinguished from Solids and Gases.— A *fluid* is a substance whose particles are moved among each other by a very slight force. In solid bodies the particles are held by the force of cohesion in fixed relations to each other; hence such bodies retain their form in spite of gravity or other small forces exerted upon them. If a solid be reduced to the finest powder, still each grain of the powder is a solid body, and its atoms are held together in a determinate shape. A pulverized solid, if piled up, will settle by the force of gravity to a certain inclination, according to the smallness and smoothness of its particles, while a liquid will not rest till its surface is horizontal.

Fluids are of two kinds, liquids and gases. In a *liquid*, there is a perceptible cohesion among its particles; but in a *gas*, the particles mutually repel each other. These fluids are also distinguished by the fact that liquids cannot be compressed except in a very slight degree, while the gases are very compressible. A force of 15 pounds on a square inch, applied to a mass of water, will compress it only about .000046 of its volume, as is shown by an instrument devised by Oersted. But the same force applied to a quantity of air of the usual density at the earth's surface will reduce it to one-half of its former volume.

173. Transmitted Pressure.—It is an observed property of fluids that a force which is applied to one part is transmitted undivided to all parts. For instance, if a piston A (Fig. 123) is pressed upon the water in the vessel $A\ D\ C$ with a force of *one pound*, every other piston of the same size, as B, C, D, or E, receives a pressure of *one pound* in addition to the previous pressure of the water itself. Hence the whole amount of bursting pressure exerted within the vessel by the weight

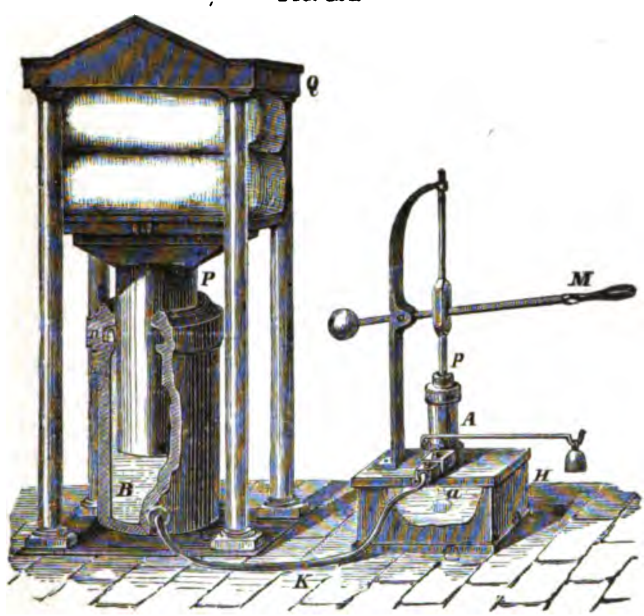
FIG. 123.



upon *A* equals as many pounds as there are portions of surface equal to the area of A. And if the pressure is increased till the vessel bursts, the fracture is as likely to occur in some other part as in that toward which the force is directed.

174. The Hydraulic Press.—An important application of the principle of transmitted pressure occurs in Bramah's hydraulic press, represented in Fig. 124. The walls of the cylinder and

FIG. 124.



reservoir are partly removed, to show the interior. *A* is a small forcing pump, worked by the lever *M*, by which water is raised in the pipe *a* from the reservoir *H*, and driven through the tube *K* into the cylinder *B*, where it presses up the piston *P*, and the iron plate on the top of it, against the substance above. At each downward stroke of the small piston *p*, a quantity of water is transferred to the cylinder *B*, and presses up the large piston with a force as many times greater than that exerted on the small one as the under surface of *P* is greater than that of *p* (Art 173). If the diameter of *p* is *one* inch, and that of *P* is *ten* inches, then any pressure on *p* exerts a pressure 100 times as great on *P*. The lever *M* gives an additional advantage. If the distances from the fulcrum to the rod *p* and to the hand are as 1 : 5; this ratio compounded with the other, 1 : 100, gives the ratio of power at *M* to

the pressure at Q as 1 : 500; so that a power of 100 lbs. exerts a pressure of 50000 lbs.

This machine has the special advantage of working with a small amount of friction. It is used for pressing paper and books, packing cotton, hay, &c.; also for testing the strength of cables and steam-boilers. It has been sometimes employed to raise great weights, as, for instance, the tubular bridge over the Menai straits; the two portions, after being constructed at the water level, were raised more than 100 feet to the top of the piers, by two hydraulic presses. The weight of each length lifted at once was more than 1800 tons.

The relation of power to weight in the hydraulic press is in accordance with the principle of virtual velocities (Art. 141). For, while a given quantity of water is transferred from the smaller to the larger cylinder, the velocity of the large piston is as much less than that of the small one as its area is greater. But we have seen that the pressures are directly as the areas. Therefore, in this as in other machines, the intensities of the forces are inversely as their virtual velocities.

Ex. 1. A press of the same form as in Fig. 124 has a piston whose cross section is one sq. ft.; the feed-pump piston is 2 sq. in. cross section, and stroke 6 inches. The lever has a short arm of 1 ft. and long arm of 4 ft. (measured from fulcrum in each case), Find the greatest pressure that can be produced by a man who exerts a force of 174 lbs., friction and difference of level of the liquid in the cylinders being disregarded. *Ans.* 50112 lbs.

Ex. 2. How many strokes of the pump will it take to raise the press piston one foot in the last example? *Ans.* 144.

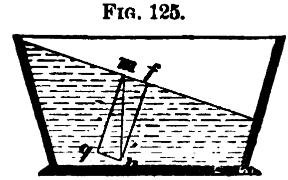
175. Equilibrium of a Fluid.—In order that a fluid may be at rest,

1. *The pressures at any one point must be equal in all directions.*

2. *The surface must be perpendicular to the resultant of the forces which act upon it.*

Both of these conditions result from the mobility of the particles. It is obvious that the first must be true, since, if any particle were pressed more in one direction than another, it would move in the direction of the greater force, and therefore not be at rest, as supposed.

FIG. 125.



In order to show the truth of the second condition, let $m\,p$ (Fig. 125) represent the resultant of the forces which act on the fluid. Then, if the

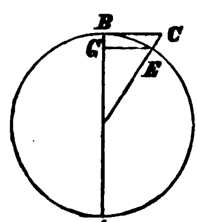
surface is not perpendicular to $m\,p$, that force may be resolved into $m\,q$ perpendicular to the surface, and $m\,f$ parallel to it. The latter, $m\,f$, not being opposed, the particles move in that direction.

As gravity is the principal force which acts on all the particles, the surface of a fluid at rest is ordinarily *level*, that is, perpendicular to a vertical or plumb line. If the surface is of small extent, it is sensibly a plane, though it is really curved, because the vertical lines, to which it is perpendicular, converge toward the centre of the earth.

176. The Curvature of a Liquid Surface.—The earth being 7912 miles in diameter, a distance of 100 feet on its surface subtends an angle of about one second at the centre, and therefore the levels of two places 100 feet apart are inclined one second to each other.

The amount of depression for moderate distances is found by the formula, $d = \frac{2}{3}\,L^2$. in which d is the depression in feet, and L the length of arc in miles. Let $B\,E$ (Fig. 126) be a small arc of a great circle on the earth; then $C\,E$ is the depression. As $B\,E$ is small, its chord may be considered equal to the arc, and $B\,G$ equal to the depression. But $B\,G : B\,E :: B\,E : B\,A$; that is, $d : L :: L : 7912$; or $d = \frac{L^2}{7912}$.

Fig. 126.



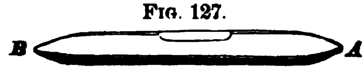
In order to express d in feet, while the other lines are in miles, we have

$$d = \frac{L^2 \times 5280^2}{7912 \times 5280} = \frac{L^2 \times 5280}{7912} = \tfrac{2}{3}\,L^2, \text{ very nearly.}$$

This gives, for one mile, $d = 8$ inches; for two miles, $d = 2$ ft. 8 in.; and for 100 miles, $d = 6667$ ft., &c. If a canal is 100 miles long, each end is more than a mile below the tangent to the surface of the water at the other end.

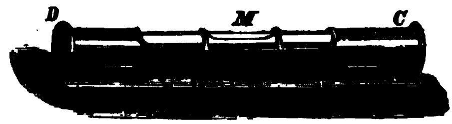
177. The Spirit Level.—Since the surface of a liquid at rest is level, any straight line which is placed parallel to such a surface is also level. Leveling instruments are constructed on this principle. The most accurate kind is the one called the *spirit* level. Its most essential part is a glass tube, $A\,B$ (Fig. 127), nearly filled with alcohol (because water would be liable to freeze), and hermetically sealed. The tube having a little convexity upward from end to end, though so slight as not to be

Fig. 127.



visible, the bubble of air moves to the highest part, and changes its place by the least inclination of the tube. The tube is so connected with a straight bar of wood or metal, as $D\ C$ (Fig. 128), or for nicer purposes, with a telescope, that the bubble is at the middle M when the bar or the axis of the telescope is exactly level. The tube usually has graduation lines upon it for adjusting the bubble accurately to the middle.

FIG. 128.



178. Pressure as Depth.—From the principle of equal transmission of force in a fluid, it follows that, if a liquid is uniformly dense, its pressure on a given area varies as the perpendicular depth, whatever the form or size of the reservoir. Let the vessel $A\ B\ C\ D$ (Fig. 129), having the form of a right prism, be filled with water, and imagine the water to be divided by horizontal planes into strata of equal thickness. If the density is everywhere the same, the weights of these strata are equal. But the pressure on each stratum is the sum of the weights of all the strata above it. Therefore, in this case, the pressure varies as the depth.

FIG. 129.

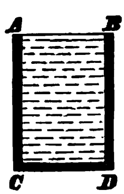


FIG. 130.

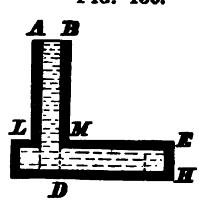
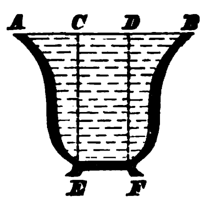


FIG. 131.



But let the reservoir (Fig. 130) contain water. The pressure of the column $A\ B\ L\ M$ is transmitted equally in every direction (Art. 173). If the area of the section $L\ M$ is one sq. inch and the weight of the column $A\ L$ is one pound, then every square inch of the side $E\ H$ will receive a pressure of one pound, on account of the column $A\ L$, in addition to the pressure it sustains from the contained water. So also every square inch of the bottom $D\ H$ will sustain an added pressure of 1 lb., and also every square inch of the top $M\ E$ will sustain an *upward* pressure of 1 lb. That is to say, the added pressure upon every part of the containing vessel $L\ E\ H\ D$, whose area equals the area of the base of the column $A\ L\ M\ B$, is equal to the weight of that column.

Again, if the base is smaller than the top, as in the vessel

$A\ B\ E\ F$ (Fig. 131), then the pressure on $E\ F$ equals only the weight of the column $C\ D\ E\ F$. The water in the surrounding space $A\ C\ E$, $B\ D\ F$, simply serves as a vertical wall to balance the lateral pressures of the central column.

If the surface pressed upon is oblique or vertical, then the points of it are at unequal depths; in this case, the depth of the area is understood to be the *average* depth of all its parts; that is, the depth of its centre of gravity.

If the fluid were compressible, the lower strata would be more dense than the upper ones, and therefore the pressure would increase at a faster rate than the depth.

The following experiment will show that the pressure of a liquid upon a given base is due to the depth of the liquid and is independent of the volume. Bend a glass tube $A\ B\ C$, as shown in Fig. 132 (a), and attach a cup $D\ E$, into which may be screwed

FIG. 132 (a).

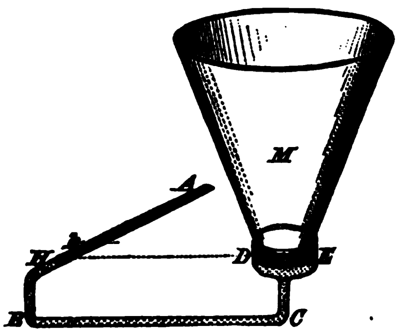
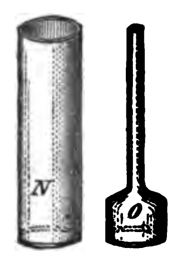


FIG. 132 (b).



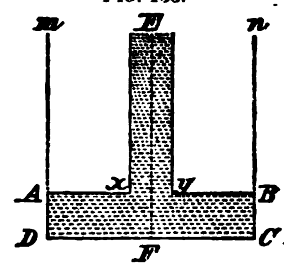
various shaped receivers M, N, O, &c. Into the cup $D\ E$ pour mercury, which will stand at a level, say $H\ D\ E$. Now screw into the cup any one of the receivers, as M, and pour in water to any desired height. The mercury will be depressed in the cup $D\ E$ by the pressure of the water, and will rise in $A\ B$ to some point h. Now remove the vessel M and substitute in succession each of the others, filling to the same height as before. The mercury in each case will rise to the point h, showing that the pressure upon the area of mercury in the cup $D\ E$ is the same in all cases, for a given height of liquid.

179. Hydrostatic Paradox.—To guard against a possible misapprehension in this connection, the student must be cautioned to distinguish between the *pressure* upon the bottom and the *weight* of the contained liquid.

In the vessel $A\ B\ C\ D$ (Fig. 133), the *pressure* upon the bot-

tom is equal to the area of the base $D\,C$ in square inches multiplied by the weight of a column of water of one sq. inch cross-section and height $E\,F$, which product is equal to the weight of a column of cross-section $D\,C$ and height $E\,F$, or the whole volume $m\,D\,C\,n$. But the *weight* of the contained water is less than this, as shown by the figure.

FIG. 133.



To illustrate, suppose area of base $D\,C = 12$ sq. inches, $A\,D = 1$ inch, $E\,F = 11$ inches, and $x\,y = 1$ sq. inch, and call the weight of one cubic inch of water w. Then the *pressure* upon the base $D\,C = 12 \times 11 \times w = 132\ w$. The *weight* of the liquid $= 12 \times 1 \times w + 10 \times 1 \times w = 22\ w$.

The *downward* pressure upon the base is, as above, 132 w. The *upward* pressure upon the upper base $A\,B$ is equal to the area of the *ring* $A\,B$ multiplied by the height $m\,A$, or equal to $11 \times 10 \times w = 110\ w$. Downward pressure minus upward pressure = weight.

$$132\ w - 110\ w = 22\ w, \text{ as before.}$$

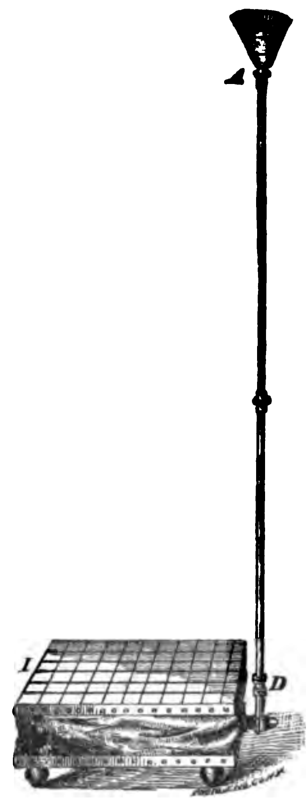
180. Amount of Pressure in Water.—One cubic foot of water weighs 1000 ounces, or 62.5 pounds avoirdupois. Therefore, the pressure on *one square foot*, at the depth of *one foot*, is 62.5 pounds. From this, as the *unit* of hydrostatic pressure, it is easy to determine the pressures on all surfaces, at all depths; for it is obvious that, when the depth is the same, the pressure varies as the surface pressed upon; and it has been shown that, on a given surface, the pressure varies as the depth of its centre of gravity; it therefore varies as the product of the two. Let p = pressure; a = area pressed upon; and d = the depth of its centre of gravity; then $p = a\,d \times 62.5$.

Depth.	*Lbs. per sq. ft.*	*Depth.*	*Lbs. per sq. ft.*
1 ft.	62.5	100 ft	6,250
10	625	1 mile	330,000
16	1000	5 miles	1,650,000

From the above table it may be inferred that the pressure on a square foot in the deepest parts of the ocean must be not far from two millions of pounds; for the depth in some places is more than five miles, and sea-water weighs 64.37 pounds, instead of 62.5 pounds. A brass vessel full of air, containing only a pint, and whose walls were one inch thick, has been known to be crushed in by this great pressure, when sunk to the bottom of the ocean.

Owing to the increase of pressure with depth, there is great difficulty in confining a high column of water by artificial structures. The strength of banks, dams, flood-gates, and aqueduct pipes, must increase in the same ratio as the perpendicular depth from the surface of the water, without regard to its horizontal extent.

FIG. 134.



181. Column of Water whose Weight Equals the Pressure.—A convenient mode of conceiving readily of the amount of pressure on an area, in any given circumstances, is this: consider the area pressed upon to form the horizontal base of a hollow prism; let the height of the prism equal the average depth of the area; and then suppose it filled with water. The weight of this column of water is equal to the pressure. For the contents of the prism (whose base $= a$, and its height $= d$), $= a\,d$; and the weight of the same $= a\,d \times 62.5$ lbs.; which is the same expression as was obtained above for the pressure.

On the bottom of a cubical vessel full of water, the pressure equals the *weight* of the water; on each side of the same the pressure is *one-half* the weight of the water; hence, on all the five sides the pressure is *three times* the weight of the water; and if the top were closed, on which the pressure is zero, the pressure on the six sides is the same, three times the weight of the water.

182. Illustrations of Hydrostatic Pressure.—A vessel may be formed so that both its base and height shall be great, but its cubical contents small; in which case, a great pressure is produced by a small quantity of water. The hydrostatic bellows is an example. In Fig. 134 the weight which can be sustained on the lid *D I* by the column *A D* is equal to that of a prism or cylinder of water, whose base is *D I*, and its height *D A*. It is immaterial how shallow is the stratum of water on the base, or how slender the tube *A D*, if greater than a capillary size.

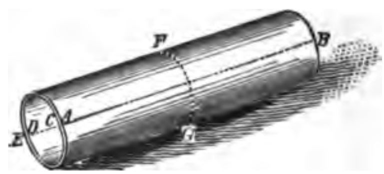
In like manner, a cask, after being filled, may be burst by an additional pint of water; for, by screwing a long and slender pipe

into the top of the cask, and filling it with water, the pressure is easily made greater than the strength of the cask can bear.

183. Determination of Thickness of Cylinder.—To determine the thickness of plate required in a cylindrical vessel that it may sustain a given pressure, we assume that the bursting results from tearing asunder the material of the plate.

Let $A\ B\ E$ (Fig. 135) represent the cylindrical vessel; $E\ D\ C\ A\ B$ a longitudinal section through the axis: put $a = A\ B =$ length in inches, $2\ r = C\ D =$ internal diameter in inches, $e = A\ C = D\ E =$ thickness of plate in inches, $T =$ tenacity of the material in lbs. per sq. inch; then

Fig. 135.



$a \times e \times T =$ strength, or resistance to tearing apart, of section $B\ A\ C$. As there are two such sections which resist the internal pressure the total strength through the section $E\ D\ C\ A\ B$, is $2\ a \times e \times T$.

Call the internal pressure in lbs. per sq. inch P. The total bursting pressure through the section $C\ D$, acting upward and downward to cause separation in that plane, is equal to the area multiplied by the pressure per sq. inch, or = rectangle $2\ r \times a \times P$. But at the moment of rupture these two must be equal, therefore

$2\ a\ e\ T = 2\ r \times a \times P$, whence $e = \frac{r \times P}{T}$ which gives the thickness when the internal diameter, the tenacity and the pressure are known. The longitudinal section through the axis is the weakest longitudinal section that can be taken, hence we need consider no other.

To determine the thickness to withstand rupture through the transverse section $G\ F$, we have, area of section of material through $G\ F = \pi\ (r + e)^2 - \pi\ r^2 = \pi\ e\ (e + 2\ r)$, and the tenacity of section $= \pi\ e\ (e + 2\ r)\ T$.

The bursting pressure upon the plane through $G\ F$, exerted upon the heads of the cylinder, $= \pi \times r^2 \times P$. These being equal we have,

$$\pi\ e\ (e + 2\ r)\ T = \pi\ r^2\ P,$$

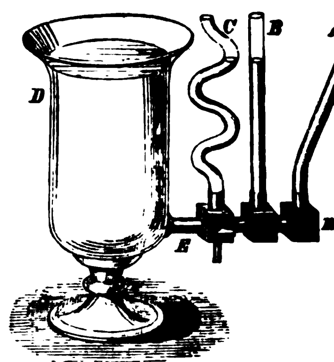
$$2 e \left(\frac{e}{2\ r} + 1\right) T = r\ P;$$

neglecting $\frac{e}{2\ r}$, which will usually be a small fraction, we get $e = \frac{r\ P}{2\ T}$.

Comparing this with the previous result we find that the transverse section requires only half the thickness of material, for a given pressure, which is required by the longitudinal section, hence this section need not be considered in determining the thickness.

184. The Same Level in Connected Vessels.—In tubes or reservoirs which communicate with each other, water will rest only when its surface is at the same level in them all. If water is poured into D (Fig. 136), it will rise in the vertical tube B, so as to stand at the same level as in D. For, the pressure toward the right on any cross-section E of the horizontal pipe $m\ n$ equals the product of its area by its depth below D. So the pressure on the same section towards the left equals the product of its area by its depth below B. But these pressures are equal, since the liquid is at rest.

FIG. 136.

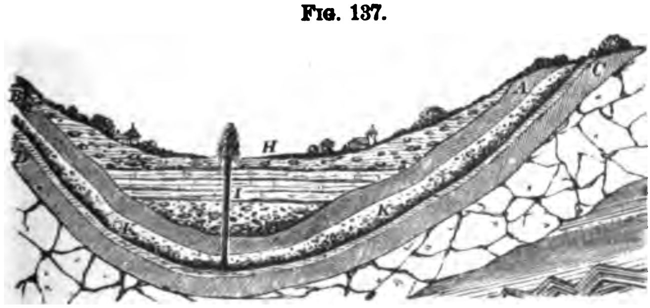


Therefore E is at equal depths below B and D; in other words, B and D are on the same level. The same reasoning applies to the irregular tubes A and C, and to any others, of whatever form or size.

Water conveyed in aqueducts, or running in natural channels in the earth, will rise just as high as the source, but no higher.

Artesian wells illustrate the same tendency of water to rise to its level in the different branches of a tube. When a deep boring is made in the earth, it may strike a layer or channel of water which descends from elevated land, sometimes very distant. The pressure causes it to rise in the tube, and often throws it many feet above the surface. Fig. 137 shows an artesian well, through

FIG. 137.



which is discharged the water that descends in the porous stratum $K\ K$, confined between the strata of clay $A\ B$ and $C\ D$.

A tube driven to the water bed anywhere between A or B and the lowest point in the diagram, might also bring water to the surface if the flow *below* the end of the tube were sufficiently obstructed by friction ; hence an artesian well might be successfully driven when the inclination of the water bed is wholly in one direction.

185. Centre of Pressure.—The centre of pressure of any surface immersed in water is that point through which passes the resultant of all the pressures on the surface. It is the point, therefore, at which a single force must be applied in order to counterbalance all the pressures exerted on the surface. If the surface be a plane, and horizontal, the centre of pressure coincides with the centre of gravity, because the pressures are equal on every part of it, just as the force of gravity is. But if the plane surface makes an angle with the horizon, the centre of pressure is lower than the centre of gravity, since the pressure increases with the depth. For example, if the vertical side of a vessel full of water is rectangular, the centre is *one-third* of the distance from the middle of the base to the middle of the upper side. If triangular, with its lower side horizontal, the centre of pressure is *one-fourth* of the distance from the middle of the base to the vertex. If triangular, with the top horizontal, the centre of pressure is *half* way up on the bisecting line.

[See Appendix for calculations of the place of the centre of pressure.]

186. The Loss of Weight in Water.—When a body is immersed in water, it suffers a pressure on every side, which is proportional to the depth. Opposite components of lateral pressures, being exerted on surfaces at the same depth, balance each other ; but this cannot be true of the vertical pressures, since the top and bottom of the body are at unequal depths. The upward pressure on the bottom exceeds the downward pressure on the top ; and this excess constitutes the *buoyant power* of a fluid, which causes a loss of weight.

A body immersed in water loses weight equal to the weight of water displaced.

For before the body was immersed, the water occupying the same space was exactly supported, being pressed upward more than downward by a force equal to its own weight. The weight of the *body*, therefore, is diminished by this same difference of pressures, that is, by the weight of the displaced water.

To show this experimentally, suspend a solid cylinder *A* (Fig. 138) below a hollow cylinder *B*, into which it will fit with great nicety; attach both to the arm of a balance and carefully counterpoise them; now pour water, or any other liquid, into the beaker *C* until it is full, and the equilibrium will be destroyed, the end of the beam *D* rising. Fill the cylinder *B* with the same liquid, and when it is exactly full, the cylinder *A* will be found to be submerged exactly to its upper edge, thus showing that the buoyancy of the liquid in this case is counteracted by a volume of the same liquid equal to the volume of the submerged body.

Fig. 138.

On the supposition of the complete incompressibility of water, this loss is the same at all depths, because the weight of displaced water is the same. As water, however, is slightly compressible, its buoyant power must increase a little at great depths. Calling the compression .000046 for one atmosphere (= 34 feet of water), the bulk of water at the depth of a mile is reduced by about $\frac{1}{140}$, and its specific gravity increased in the same ratio; so that, *possibly*, a body might sink near the surface, and float at great depths in the ocean. But this is not *probable* in any case, since the same compressing force may reduce the volume of the solid as much as that of the water. And, furthermore, the increase of density by increased depth is so slow, that even if solids were incompressible, most of those which sink at all would not find their floating place within the greatest depths of the ocean. For example, a stone twice as heavy as water must sink 100 miles before it could float.

187. Equilibrium of Floating Bodies.—If the body which is immersed has the same density as water, it simply loses its whole weight, and remains wherever it is placed. But if it is less dense than water, the excess of upward pressure is more than sufficient to support it; it is, therefore, raised to the surface, and comes to a state of equilibrium after partly emerging. In order that a floating body may have a stable equilibrium, the three following conditions must be fulfilled:

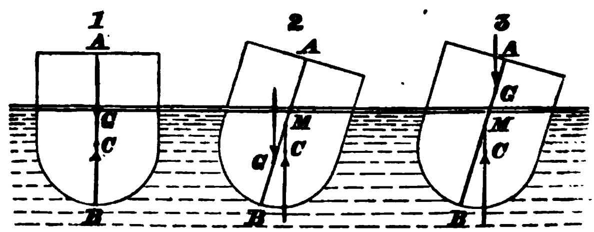
1. *It displaces an amount of water whose weight is equal to its own.*

2. *The centre of gravity of the body is in the same vertical line with that of the displaced water.*

3. *The metacenter is higher than the centre of gravity of the body.*

The reason for the *first* condition is obvious; for both the body and the water displaced by it are sustained by the same upward pressures, and therefore must be of equal weight.

FIG. 139.



That the *second* is true, is proved as follows: Let C (Fig. 139, 1) be the centre of gravity of the displaced water, while that of the body is at G. Now the fluid, previous to its removal, was sustained by an upward force equal to its own weight, acting through its centre of gravity C; and the same upward force now acts upon the floating body through the same point. But the body is urged downward by gravity in the direction of the vertical line $A\ G\ B$. Were those two forces exactly opposite and equal, they would keep the body at rest; but this is the case only when the points C and G are in the same vertical line; in every other position of these points, the two parallel forces tend to turn the body round on a point between them.

188. The Metacenter.—To understand the *third* condition, the metacenter must be defined. A floating body assumes a position such that the line through the centres of gravity of the body and of the displaced water shall be vertical; now, regard this line so determined as fixed with respect to the body, moving with it to any degree of inclination; then move the body so that this line shall make an indefinitely small angle with its vertical position; the intersection of the line as now placed with the vertical through the new centre of gravity of the displaced water is called the metacenter. When the centre of gravity of the body G is *lower* than the metacenter, as in Fig. 139, 2, the parallel forces, downward through G and upward through C, revolve the body back to its position of equilibrium, which is then called a stable equilibrium. But if the centre of gravity of the body is

higher than the metacenter, as in Fig. 139, 3, the rotation is in the opposite direction, and the body is upset, the equilibrium being unstable. Once more, if the centre of gravity of the body is *at* the metacenter, the body rests indifferently in any position, as, for example, a sphere of uniform density. The equilibrium in this case is called neutral.

If only the first condition is fulfilled, there is *no* equilibrium; if only the first and second, the equilibrium is *unstable;* if all the three, the equilibrium is *stable.*

In accordance with the third condition, it is necessary to place the heaviest parts of a ship's cargo in the bottom of the vessel, and sometimes, if the cargo consists of light materials, to fill the bottom with stone or iron, called *ballast*, lest the masts and rigging should raise the centre of gravity too high for stability. On the same principle, those articles which are prepared for life-preservers, in case of shipwreck, should be attached to the upper part of the body, that the head may be kept above water. The danger arising from several persons standing up in a small boat is quite apparent; for the centre of gravity is elevated, and liable to become higher than the metacenter, thus producing an unstable equilibrium.

189. Floating in a Small Quantity of Water.—As pressure on a given surface depends solely on the depth, and not at all on the extent or quantity of water, it follows that a body will float as freely in a space slightly larger than itself as on the open water of a lake. For instance, a ship may be floated by a few hogsheads of water in a dock whose form is adapted to it. In such a case, it cannot be literally true that the displaced water weighs as much as the vessel, when *all* the water in the dock may not weigh a hundredth part as much. The expression "displaced water" means the amount which would fill the place occupied by the immersed portion of the body. An experiment illustrative of the above is, to float a tumbler within another by means of a spoonful of water between.

190. Floating of Heavy Substances.—A body of the most dense material may float, if it has such a form given it as to exclude the water from the upper side, till the required amount is displaced. Ships are built of iron, and laden with substances of greater specific gravity than water, and yet ride safely on the ocean. A block of any heavy material, as lead, may be sustained by the upward pressure beneath it, provided the water is excluded from the upper side by a tube fitted to it by a water-tight joint.

191. Specific Gravity.—The weight of a body compared

with the weight of the same volume of the standard, is called its *specific gravity.*

Distilled water at about 4° C. (the temperature of its greatest density) is the standard for solids and liquids, and air, at 760 mm. and 0°, for gases. Therefore the specific gravity of a body equals its weight divided by the weight of an equal volume of water or air, as the case may be.

192. Methods of Finding Specific Gravity.—

1. REGULAR SOLID.—*Divide the weight in grams by the volume in cubic centimeters.*

Inasmuch as a c.cm. of water weighs one gram, the volume of a body in c.cm.'s would be the same as the weight of the same volume of water.

2. SOLID—HEAVIER THAN WATER.—*Divide its weight in air by its loss of weight in water.*

A body weighed in water is lighter by the weight of an equal volume of water.

3. SOLID—LIGHTER THAN WATER.—*Weigh the body in air. Attach a sinker beneath the body and make two weighings—one with the body in air and the sinker under water, the other with both body and sinker under water. Divide the weight in air by the difference of the last two weights.*

A cork, with a lead sinker attached and submerged, would weigh more than when both cork and sinker were submerged, by an amount equal to the weight of the water displaced by the cork.

In making accurate determinations of the specific gravity of solids, care must be taken to remove any adhering bubbles of air before the weighing in water is made.

If the body whose specific gravity is required be soluble in water, its specific gravity must be determined with reference to some liquid which will not dissolve it, such as alcohol, turpentine, a saturated solution of the substance itself, &c., and then the specific gravity so obtained must be multiplied by the specific gravity of the liquid used, as compared with water.

4. A LIQUID. — *Find the loss which a body sustains weighed in the liquid and then in water, and divide the first loss by the second.*

For the first loss equals the weight of the displaced liquid, and the second that of the displaced water; and the volume in each case is the same, namely, that of the body weighed in them.

But the specific gravity of a liquid may be more directly obtained by measuring equal volumes of it and of water in a flask.

and finding the weight of each. Then the weight of the liquid divided by that of the water is the specific gravity required.

Flasks for the purpose are made with carefully ground stoppers through which is pierced a fine hole so that in inserting the stopper there may be an overflow through the hole, after which the flask having been carefully wiped off, it is ready for weighing.

193. The Hydrometer, or Areometer.—In commerce and the arts, the specific gravities of substances are obtained in a more direct and sufficiently accurate way, by instruments constructed for the purpose. The general name for such instruments is the *hydrometer*, or *areometer*. But other names are given to such as are limited to particular uses; as, for example, the *alcoömeter* for alcohol, and the *lactometer* for milk. The hydrometer, represented in Fig. 140, consists of a hollow ball, with a graduated stem. Below the ball is a bulb containing mercury, which gives the instrument a stable equilibrium when in an upright position. Since it will descend until it has displaced a quantity of the fluid equal in weight to itself, it will of course sink to a greater depth if the fluid is lighter. From the depths to which it sinks, therefore, as indicated by the graduated stem, the corresponding specific gravities are estimated.

FIG. 140.

The sensibility of instruments of this class is increased by diminishing the diameter of the stem.

FIG. 141.

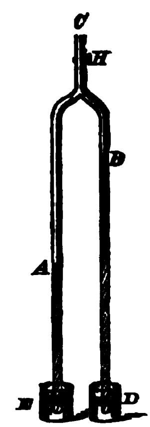
Nicholson's hydrometer (Fig. 141) is the most useful of this class of instruments, since it may be applied to finding the specific gravities of solid as well as liquid bodies. In addition to the hollow ball of the common hydrometer, it is furnished at the top with a pan A for receiving weights, and a cavity beneath for holding the substance under trial. The instrument is so adjusted that when 10 grams are placed in the pan, the instrument sinks in distilled water at the temperature of 4° C. to a fixed mark, 0, on the stem. Calling the weight of the instrument W, the weight of displaced water is $W + 10$.

To find the specific gravity of a *liquid*, place in the pan such a weight w as will just bring the mark to the surface. Then the weight of the liquid displaced is $W + w$. But its volume is equal to that of the displaced water. Therefore its specific gravity is $\frac{W + w}{W + 10}$.

To find the specific gravity of a *solid*, place in the pan a fragment of it weighing less than 10 grams, and add the weight w required to sink the mark to the water-level. Then the weight of the substance in air is $10 - w$. Remove the substance to the cavity at the bottom of the instrument, and add to the weight in the pan a sufficient number of grams w' to sink the mark to the surface. Then w' is the *loss* of weight in water; therefore, $\frac{10 - w}{w'}$ is the specific gravity of the substance.

194. Specific Gravity of Liquids by Means of Heights.—This method depends upon the fact that the heights at which columns of liquids will be sustained by any given atmospheric pressure are inversely as their specific gravities.

Fig. 142.



Arrange two glass tubes A and B (Fig. 142) connected at the top with a common outlet C, their lower ends being immersed in the liquids contained in the beakers E and D. Exhaust the air by the outlet C till the liquids rise to any desired height, say to A and B, and suppose the height of the column A, measured from the surface of the liquid in beaker E to be one-half that of the column B measured from the surface in beaker D; then the specific gravity of liquid E is twice that of liquid D.

This method gives only approximate results, depending upon the fineness of division of the scales used, corrected for capillarity of the tubes.

195. Table of Specific Gravities.—An accurate knowledge of the specific gravities of bodies is important for many purposes of science and art, and they have therefore been determined with the greatest possible precision. The heaviest of ordinary substances is *platinum*, whose specific gravity, when compressed by rolling, is 22, water being 1; and the lightest is *hydrogen*, whose specific gravity is = .069, common air being 1. Now, as water is about 800 times as heavy as air, it is (800 ÷ .069 =) 11,594 times as heavy as hydrogen. Therefore platinum is about (11,594 × 22 =) 255,068 times as heavy as hydrogen. Between

these limits, 1 and 255,068, there is a wide range for the specific gravities of other substances. The values for some substances are given in the following tables:

Water at 4° C. = 1.

Aluminium	2.6
Copper	8.5–8.9
German silver	8.5
Glass	2.5–3.5
Gold	19.3
Iron	7.1–7.8
Lead	11.3
Mercury	13.6
Platinum	22.0
Silver	10.4
Wood	0.2–1.2
Zinc	7.2

[*At* 15° *C.*]

Alcohol	0.7938
Chloroform	1.499
Ether	0.720
Glycerine	1.260
Sulphuric acid (conc.)	1.838

Air = 1.

Carbonic Acid	1.5200
Hydrogen	0.0692
Nitrogen	0.9701
Oxygen	1.1052

196. Floating.—The human body, when the lungs are filled with air, is lighter than water, and but for the difficulty of keeping the lungs constantly inflated, it would naturally float. With a moderate degree of skill, therefore, swimming becomes a very easy process, especially in salt water. When, however, a man plunges, as divers sometimes do, to a great depth, the air in the lungs becomes compressed, and the body does not rise except by muscular effort. The bodies of drowned persons rise and float after a few days, in consequence of the inflation occasioned by putrefaction.

As rocks are generally not much more than twice as heavy as water, nearly half their weight is sustained while they are under water; hence, their weight seems to be greatly increased as soon as they are raised above the surface. It is in part owing to their diminished weight that large masses of rock are transported with

great facility by a torrent. While bathing, a person's limbs feel as if they had nearly lost their weight, and when he leaves the water, they seem unusually heavy.

197. To find the Magnitude of an Irregular Body.—It would be a long and difficult operation to find the exact contents of an irregular mineral by direct measurement. But it might be found with facility and accuracy by weighing it in air, and then finding its loss of weight in water. The loss is the weight of a mass of water having the same volume. Now, as a cubic centimeter of water weighs one gram, if the weights are expressed in grams, the loss of weight equals the volume of displaced water = the volume of the mineral.

198. Cohesion and Adhesion.—What distinguishes a liquid from a solid is not its want of cohesion so much as the mobility of its particles. It is proved in many ways that the particles of a liquid strongly attract each other. It is owing to this that water so readily forms itself into drops. The same property is still more observable in mercury, which, when minutely divided, will roll over surfaces in spherical forms. When a disk of almost any substance is laid upon water, and then raised gently, it lifts a column of water after it by adhesion, till at length the edge of the fluid begins to divide, and the column is detached, not in all parts at once, but by a successive rupturing of the lateral surface. It is proved that the whole attraction of the liquid would be far too great to be overcome by the force applied to pull off the disk, were it not that it is encountered by little and little, at the edges of the column. But it is the cohesion of the water which is overcome in this experiment; for the upper lamina still adheres to the disk. By a pair of scales we find that it requires the same force to draw off disks of a given size, whatever the materials may be, provided they are *wet* when detached. This is what might be expected, since in each case we break the attraction between two laminæ of water. But if we use disks which are not wet by the liquid, it is not generally true that those of different material will be removed by the same force; indicating that some substances adhere to a given liquid more strongly than others.

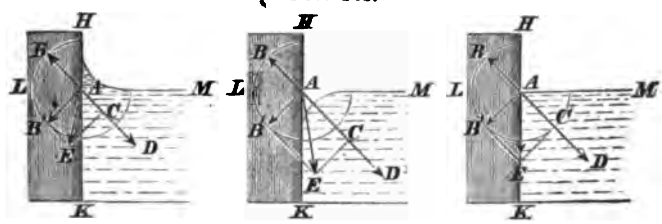
These molecular attractions extend to an exceedingly small distance, as is proved by many facts. A lamina of water adheres as strongly to the thinnest disk that can be used as to a thick one; so, also, the upper lamina coheres with equal force to the next below it, whether the layer be deep or shallow.

199. Capillary Action.—This name is given to the molecular forces, adhesion and cohesion, when they produce disturbing

effects on the surface of a liquid, elevating it above or depressing it below the general level. These effects are called *capillary*, because most strikingly exhibited in very fine (*hair-sized*) tubes.

The liquid will be elevated in a concave curve, or depressed in a convex curve, by the side of the solid, according as the attraction of the liquid molecules for each other is less or greater than twice the attraction between the liquid and the solid.

FIG. 143.



Case 1st. Let $H\ K$ (Fig. 143, 1) and $L\ M$ be a section of the vertical side of a solid, and of the general level of the liquid. The particle A, where these lines meet, is attracted (so far as this section is concerned) by all the particles of an insensibly small quadrant of the liquid, the resultant of which attractions is in the line $A\ D$, 45° below $A\ M$. It is also attracted by all the particles in two quadrants of the solid, and the resultants are in the directions $A\ B$, 45° above, and $A\ B'$, 45° below $L\ M$.

Now suppose the force $A\ D$ to be *less than twice* $A\ B$ or $A\ B'$. Cut off $C\ D = A\ B$; then $A\ B$, being opposite and equal to $C\ D$, is in equilibrium with it. The remainder $A\ C$, being less than $A\ B'$, their resultant $A\ E$ will be directed toward the solid; and therefore the surface of the liquid, since it must be perpendicular to the resultant of forces acting on it (Art. 175), takes the direction represented; that is, concave upward.

Case 2d. Let $A\ D$ (Fig. 143, 2), the attraction of A toward the liquid particles, be *more than twice* $A\ B$, the attraction toward a quadrant of the solid. Then, making $C\ D$ equal to $A\ B$, these two resultants balance as before; and as $A\ C$ is greater than $A\ B'$, the angle between $A\ C$ and the resultant $A\ E$ is less than 45°, and A is drawn away from the solid. Therefore the surface, being perpendicular to the resultant of the molecular forces acting on it, is convex upward.

Case 3d. If $A\ D$ (Fig. 143, 3) be exactly twice $A\ B$, then $C\ D$ balances $A\ B$, and the resultant of $A\ C$ and $A\ B'$ is $A\ E$ in a vertical direction; therefore the surface at A is level, being neither elevated nor depressed.

Case 1st occurs whenever a liquid readily *wets* a solid, if brought in contact with it, as, for example, water and clean glass.

Case 2d occurs when a solid *cannot be wet* by a liquid, as glass and mercury. Case 3d is rare, and occurs at the limit between the other two; water and steel afford as good an example as any.

200. Capillary Tubes.—In fine tubes these molecular forces affect the entire columns as well as their edges. If the material of the tube can be wet by a liquid, it will raise a column of that liquid above the level, at the same time making the top of the column concave. If it is not capable of being wet, the liquid is depressed, and the top of the column is convex. The first case is illustrated by glass and water; the second by glass and mercury.

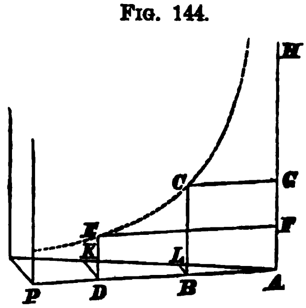
The materials being given, the distance by which the liquid is elevated or depressed varies inversely as the diameter. Therefore the product of the two is constant.

The amount of elevation and depression depends on the strength of the molecular forces, rather than on the specific gravity of the liquids. Alcohol, though lighter than water, is raised only half as high in a glass tube.

201. Parallel and Inclined Plates.—Between parallel plates a liquid rises or falls half as far as in a tube of the same diameter. This is because the sustaining force acts only on two sides of each filament, while in a tube it acts on all sides. Therefore, as in tubes the height varies inversely as the diameter, so in plates the height varies inversely as the distance between them.

If the plates are inclined to each other, having their edge of meeting perpendicular to the horizon, the surface of a liquid rising between them assumes the form of a *hyperbola*, whose branches approach the vertical edge, and the water-level, as the asymptotes of the curve. This results from the law already stated, that the height varies inversely as the distance between the plates. Let the edge of meeting, *A H* (Fig. 144), be the axis of ordinates, and the line in which the level surface of the water intersects the glass, *A P*, the axis of abscissas. Let *B C*, *D E*, be any ordinates, and *A B*, *A D*, their abscissas, and *B L*, *D K*, the distances between the plates. By the law of capillarity, the heights *B C*, *D E*, are inversely as *B L*, *D K*. But, by the similar triangles, *A B L*, *A D K*, *B L*, *D K*, are as *A B*, *A D*; therefore, *B C*, *D E*, are inversely as *A B*, *A D*; and this is a property of the hyperbola with reference to the centre and asymptotes, that the ordinates are inversely as the abscissas.

FIG. 144.



202. Effects of Capillarity on Floating Bodies.—Some cases of apparent attractions and repulsions between floating bodies are caused by the forms which the liquid assumes on the sides of the bodies. If two balls raise the water about them, and are so near to each other that the concave surfaces between them meet in one, they immediately approach each other till they touch; and then, if either be moved, the other will follow it. The water, which is raised and hangs suspended between them, draws them together.

Again, if each ball depresses the water around it, they will also move to each other, and be held together, so soon as they are near enough for the convex surfaces to meet. In this case, they are not pulled, but pushed together by the hydrostatic pressure of the higher water on the outside.

Once more, if one ball raises the water, and the other depresses it, and they are brought so near each other that the curves meet, they immediately move apart, as if repelled. For now the equilibrium is destroyed in a way just the reverse of the preceding cases. The water between the balls is too high for that which depresses, and too low for that which raises the water, so that the former is pushed away, and the latter is drawn away.

The first case, which is by far the most common, explains the fact often observed, that floating fragments are liable to be gathered into clusters; for most substances are capable of being wet, and therefore they raise the water about them.

203. Illustrations of Capillary Action.—It is by capillary action that a part of the water which falls on the earth is kept near its surface, instead of sinking to the lowest depths of the soil. This force aids the ascent of sap in the pores of plants. It lifts the oil between the fibres of the lamp-wick to the place of combustion. Cloth rapidly imbibes moisture by its numerous capillary spaces, so that it can be used for wiping things dry. If paper is not *sized*, it also imbibes moisture quickly, and can be used as *blotting-paper;* but when its pores are filled with sizing, to fit it for writing, it absorbs moisture only in a slight degree and the ink which is applied to it must dry by evaporation.

The great strength of the capillary force is shown in the effects produced by the swelling of wood and other substances when kept wet. Dry wooden wedges, driven into a groove cut around a cylinder of stone, and then occasionally wet, will at length cause it to break asunder. As the pores between the fibres of a rope run around it in spiral lines, the swelling of a rope caused by keeping it wet will contract its length with immense force.

204. Questions in Hydrostatics.—

1. The diameters of the two cylinders of a hydraulic press are *one inch* and *one foot*, respectively; before the piston descends, the column of water in the small cylinder is *two feet* higher than the bottom of the large piston. Suppose that by a screw a force of 500 lbs. is applied to the small piston; what is the whole force exerted on the large piston at the beginning of the stroke?

Ans. 72098.17 lbs.

2. A junk bottle, whose lateral surface contained 50 square inches, being let down into the sea 3000 feet, what pressure do the sides of the bottle sustain, a cubic foot of sea water weighing 64.37 lbs.? *Ans.* 67052.08+lbs.

3. What will be the apparent weight in water of a piece of rock-crystal (density 2.7) which weighs 35 grams in vacuo?

Ans. 22.04 grams.

4. A bar of aluminium (density 2.6) weighs 54.8 grams in vacuo: what will be the loss of weight when it is weighed in water?

Ans. 21.08 grams.

5. An irregular solid is found to weigh 98 grams in vacuo and 64 grams in water: what is its volume? 2, 074 ...

6. A solid cube, 4 inches in the side, is formed of a substance of specific gravity 12.5: what will its apparent weight in water be?

7. A body which weighs 24 grams in air is found to weigh 20 grams in water: what will be its apparent weight in alcohol of specific gravity 0.8? *Ans.* 20.8 grams.

8. A body which weighs 35 grams in air is found to weigh 30 grams in one fluid and 25 grams in another: what will be its weight when immersed in a mixture containing equal volumes of the two fluids? *Ans.* 27.5 grams.

9. Two bodies are in equilibrium when suspended in water from the arms of a balance: the mass of the one body is 28 and its density is 5.6; if the mass of the other is 36, what is its density?

Ans. 2.77.

10. A specific gravity bottle weighs 14.72 grams when empty, 39.74 grams when filled with water, and 44.85 grams when filled with a solution of common salt: what is the specific gravity of the solution? *Ans.* 1.204.

11. A Nicholson's hydrometer, when floating in water, required a weight of 0.15 grams to be placed upon the upper pan in order to make it sink to a fixed mark on the stem; and 5.72 grams had to be placed upon the pan in order to make it sink to the same mark in a solution of salt. If the hydrometer weighed 94.47 grams, what was the specific gravity of the solution?

12. The specific gravity of lead being 11.35; of cork, .24; of fir, .45; how much cork must be added to 60 lbs. of lead, that the united mass may weigh as much as an equal bulk of fir?

Ans. 65.8527 lbs.

CHAPTER II.

HYDRODYNAMICS.

205. Depth and Velocity of Discharge.—From an aperture which is small, compared with the breadth of the reservoir, *the velocity of discharge varies as the square root of the depth.* For the pressure on a given area varies as the depth (Art. 178). If the surface be removed, this pressure becomes a force which is measured by the momentum of the water; therefore the *momentum* varies as the depth (d). But momentum varies as the mass (m) multiplied by the velocity (v); hence $m\,v \propto d$. But it is obvious that m and v vary alike, since the greater the velocity, the greater in the same ratio is the quantity discharged. Therefore, $m^2 \propto d$, or $m \propto d^{\frac{1}{2}}$; also $v^2 \propto d$, or $v \propto d^{\frac{1}{2}}$.

Not only does the velocity vary as the square root of the depth of the orifice, but *it is equal to that acquired by a body falling through the depth.*

Let h = the height of the liquid above the orifice, and h' = the height of an infinitely thin layer at the orifice.

If this thin layer were to fall through the height h', under the action of its own weight or pressure, the velocity acquired would be $v' = \sqrt{2gh'}$ (Art. 27).

Denoting the velocity generated by the pressure of the entire column by v, we have, since velocity $\propto \sqrt{\text{depth}}$,

$$v : v' :: \sqrt{h} : \sqrt{h'}, \text{ or}$$
$$v : \sqrt{2gh'} :: \sqrt{h} : \sqrt{h'};$$
$$\therefore v = \sqrt{2gh}.$$

But $\sqrt{2gh}$ is also the velocity acquired in falling through the distance h (Art. 27).

From an orifice 16.1 feet below the surface of water, the velocity of discharge is 32.2 feet per second, because this is the velocity acquired in falling 16.1 feet; and at a depth *four* times as great.

that is, 64.4 feet, the velocity will only be doubled, that is, 64.4 feet per second.

As the velocity of discharge at any depth is equal to that of a body which has fallen a distance equal to the depth, it is theoretically immaterial whether water is taken upon a wheel from a gate at the same level, or allowed to fall on the wheel from the top of the reservoir. In practice, however, the former is best, on account of the resistance which water meets with in falling through the air.

206. Descent of Surface.—When water is discharged from the bottom of a cylindric or prismatic vessel, the surface descends with a *uniformly retarded* motion. For the velocity with which the surface descends varies as the velocity of the stream, and therefore as the square root of the depth (Art. 205). But this is a characteristic of uniformly retarded motion, that the velocity varies as the square root of the distance from the point where the motion terminates, as in the case of a body ascending perpendicularly from the earth.

The descent of the surface of water in a prismatic vessel has been used for measuring time. The *clepsydra*, or water-clock of the Romans, was a time-keeper of this description. The graduation must increase upward, as the odd numbers 1, 3, 5, 7, &c.; since, by the law of this kind of motion, the spaces passed over in equal times are as those numbers.

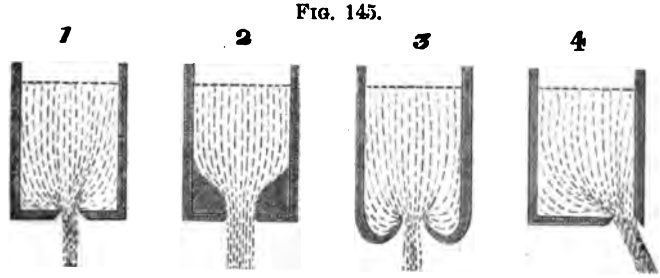
If a prismatic vessel is kept full, it discharges *twice* as much water in the same time as if it is allowed to empty itself. For the velocity in the first instance, is *uniform;* and in the second it is *uniformly retarded,* till it becomes zero. We reason in this case, therefore, as in regard to bodies moving uniformly, and with motion uniformly accelerated from rest, or uniformly retarded till it ceases (Art. 21), that the former motion is *twice* as great as the latter.

207. Discharge from Orifices in Different Situations.—Other circumstances besides *area* and *depth* of the aperture are found to have considerable influence on the velocity of discharge. Observations on the directions of the filaments are made by introducing into the water particles of some opaque substance, having the same density as water, whose movements are visible. From such observations it appears that the particles of water descend in vertical lines, until they arrive within three or four inches of the aperture, when they gradually turn in a direction more or less oblique toward the place of discharge. This convergence of the filaments extends outside of the vessel, and causes the stream to

diminish for a short distance, and then increase. The smallest section of the stream, called the *vena contracta*, is at a distance from the aperture varying from *one-half* of its diameter to the *whole*.

If water is discharged through a circular aperture in a thin plate in the bottom of the reservoir, and at a distance from the sides, as in Fig. 145, 1, the filaments form the vena contracta at a distance beyond the aperture equal to *one-half* of its diameter; the area of the section at the vena contracta is less than *two-thirds* (0.64) of the area of the aperture; this contraction also lessens the theoretical velocity by about four per cent., leaving .96 v for the final velocity; combining these two causes, it is found that for circular orifices of ½ to 6 inches in diameter, with from four to 20 feet head of water, the actual discharge is only .615 of the theoretical discharge.

FIG. 145.



If the reservoir terminates in a short pipe or ajutage, whose interior is adapted to the curvature of the filaments, as far as to the vena contracta, or a little beyond, as in Fig. 145, 2, it is found the most favorable for free discharge, which in some cases reaches 0.98 of the theoretical discharge. The stream is smooth and pellucid like a rod of glass. The most unfavorable form is that in which the ajutage, instead of being external, as in the case just described, projects inward, as in Fig. 145, 3; the filaments in this case reach the aperture, some ascending, others descending, and therefore interfere with each other. Hence the stream is much roughened in its appearance, and the flow is only 0.53 of what is due to the size of the aperture and its depth.

When the aperture is through a thin plate, the contraction of the stream and the amount of discharge are both modified by the circumstance of being near one or more sides of the reservoir. There is little or no contraction on the side next the wall of the vessel, since the filaments have no obliquity on that side; and the quantity is on that account increased. The filaments from the opposite side also divert the stream a few degrees from the perpendicular (Fig. 145, 4).

208. Friction in Pipes.—As has just been stated, an ajutage extending to or slightly beyond the vena contracta, and adapted to the form of the stream, very much increases the quantity discharged; but beyond that, the longer the pipe, the more does it impede the discharge by friction. For a given quantity of water flowing through a pipe the resistance of friction increases with the number of points with which the water comes in contact; that is, the resistance is in proportion to the wetted surface; for every particle of water in contact with the interior surface of the pipe, acts as a retarding force. Now let f be the resistance of friction in a pipe of *unit* diameter, length and velocity; then the resistance in a pipe l feet long and d feet in diameter with a unit of velocity will be $f\,d\,l$; but the quantity of water delivered by this pipe will be d^2 times that delivered by the former, in unit of time with same velocity, since areas of cross-sections are to each other as squares of their diameters; therefore for the same quantity of water delivered, the resistance of friction in the latter pipe will be $\frac{f\,d\,l}{d^2}$ or $\frac{f\,l}{d}$, that is to say, *the resistance of friction in pipes is directly as their lengths and inversely as their diameters, the velocity being constant.* In order, therefore, to convey water at a given rate through a long pipe, it is necessary either to increase the head of water or to enlarge the pipe, so as to compensate for friction.

An aqueduct should be as *straight* as possible, not only to avoid unnecessary increase of length, but because the force of the stream is diminished by all changes of direction. If there must be change, it should be a gradual curve, and not an abrupt turn. When a pipe changes its direction by an *angle*, instead of a curve, there is a useless expenditure of force; a change of 90° requires that the head of water should be increased by nearly the height due to the velocity of discharge. For instance, if the discharge is *eight* feet per second (which is the velocity due to one foot of fall), then a right angle in the pipe requires that the head of water should be increased by nearly *one* foot, in order to maintain that velocity.

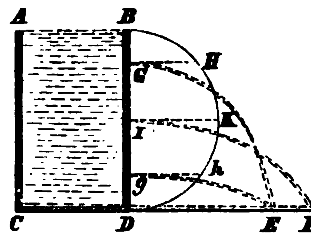
Empirical formulæ, based upon the results of experiments for the velocity of flow in pipes, and for the loss of head due to bends and angles in the pipe, are given in works treating of Practical Hydraulics. The derivation and development of such formulæ is beyond the scope of a work like this.

209. Jets.—Since a body, when projected upward with a certain velocity, will rise to the same height as that from which it must have fallen to acquire that velocity, therefore, if water issue from the side of a vessel through a pipe bent upward, it would,

were it not for the resistance of the air and friction at the orifice, rise to the level of the water in the reservoir. If water is discharged from an orifice in any other than a vertical direction, it describes a parabola, since each particle may be regarded as a projectile (Art. 47).

If a semicircle be described on the perpendicular side of a vessel as a diameter, and water issue horizontally from any point, its *range*, measured on the level of the base, equals *twice the ordinate* of that point. For, the velocity with which the fluid issues from the vessel, being that which is due to the height $B\ G$ (Fig. 146), is $\sqrt{2\,g\,.\,B\,G}$ (Art 27). But after leaving the orifice, it arrives at the horizontal plane in the time in which a body would fall freely through $G\,D$, which is $\sqrt{\frac{2\,G\,D}{g}}$. Since the horizontal motion is uniform, the space equals the product of the time by the velocity; that is, $D\,E = \sqrt{\frac{2\,G\,D}{g}} \times \sqrt{2\,g\,.\,B\,G} = 2\sqrt{B\,G\,.\,G\,D} = 2\,G\,H$, or twice the ordinate of the semicircle at the place of discharge.

FIG. 146.



The greatest range occurs when the fluid issues from the centre, for then the ordinate is greatest; and the range at equal distances above and below the centre is the same.

The remarks already made respecting pipes apply to those which convey water to the jets of fire-engines and fountains. If the pipe or hose is very long, or narrow, or crooked, or if the jet-pipe is not smoothly tapered from the full diameter of the hose to the aperture, much force is lost by friction and other resistances, especially in great velocities. If the length of hose is even *twenty* times as great as its diameter, 32 per cent. of height is lost in the jet, and more still when the ratio of length to diameter is greater than this.

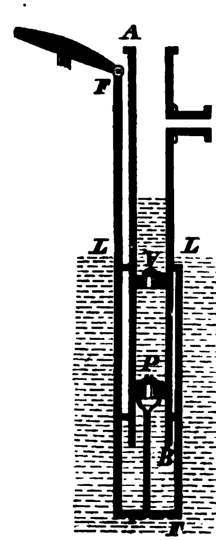
210. Rivers.—Friction and change of direction have great influence on the flow of rivers. A *dynamical equilibrium*, as it is called, exists between gravity, which causes the descent, and the resistances, which prevent acceleration at any given point, beyond a certain moderate limit; as the same quantity of water must pass every cross section of the stream in the same unit of time, under ordinary conditions, the velocity varies inversely as the area of

the cross section. The velocity in all parts of the same section, however, is not the same; it is greatest at that part of the surface where the depth is greatest, and least in contact with the bed of the stream.

To find the mean *velocity* through a given section, it is necessary to float bodies at various places on the surface, and also below it, to the bottom, and to divide the sum of all the velocities thus obtained, by the number of observations. To obtain the *quantity of water* which flows through a given section of a river, having determined the velocity as above, find next the area of the section, by taking the depth at various points of it, and multiplying the mean depth by the breadth. The quantity of water is then found by multiplying the area by the velocity.

The increased velocity of a stream during a freshet, while the stream is confined within its banks, exhibits something of the acceleration which belongs to bodies descending on an inclined plane. It presents the case of a river flowing upon the top of another river, and consequently meeting with much less resistance than when it runs upon the rough surface of the earth itself. The augmented force of a stream in a freshet arises from the simultaneous increase of the quantity of water and the velocity. In consequence of the friction of the banks and beds of rivers, and the numerous obstacles they meet with in their winding course, their velocity is usually very small, not more than three or four miles per hour; whereas, were it not for these impediments, it would become immensely great, and its effects would be exceedingly disastrous. A very slight declivity is sufficient for giving the running motion to water. The largest rivers in the world fall about five or six inches in a mile.

FIG. 147.



211. Hydraulic Pumps.—The most common pumps for raising water operate on a principle of pneumatics, and will be described under that subject.

In the *lifting pump* the water is pushed up in the pump tube by a piston placed below the water-level. In the tube A B (Fig. 147) is a fixed valve V, a little below the water-level L L, while still lower is the piston P, in which there is a valve. Both of these valves open upward. The piston is attached to a rod, which extends downward to the frame F F. This frame can be moved

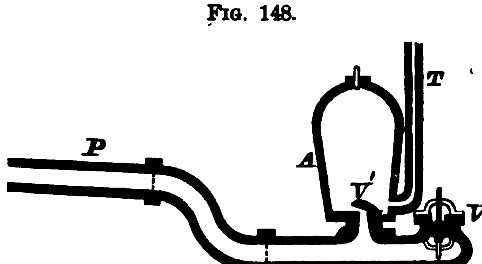
up and down on the outside of the tube by a lever. When the piston descends, the water passes through its valve by hydrostatic pressure; and when raised, it pushes the water before it through the fixed valve, which then prevents its return. In this manner, by repeated strokes, the water can be driven to any height which the instrument can bear.

The *chain pump* consists of an endless chain with circular disks attached to it at intervals of a few inches, which raise the water before them in a tube, by means of a wheel over which the chain passes; the wheel may be turned by a crank. The disks cannot fit closely in the tube without causing too great resistance; hence, a certain velocity is requisite in order to raise water to the place of discharge; and after the working of the pump ceases, the water soon descends to the level in the well.

212. Centrifugal Pumps.—Water may also be raised through small heights and in great volume by the centrifugal pump. This consists of revolving curved, hollow arms, connected with a hollow axis through which the water enters. As this axis is made to rotate in a direction contrary to the curvature of the arms the centrifugal force causes the water to leave the arms and move off in tangents; a casing drum inclosing the revolving portion forces the water to move around in a vortex till it reaches a delivery pipe entering the drum as a tangent, through which it is discharged. A high delivery requires so great velocity that the pump becomes inferior in efficiency to other forms.

213. The Hydraulic Ram.—When a large quantity of water is descending through an inclined pipe, if the lower extremity is suddenly closed, since water is nearly incompressible, the shock of the whole column is received in a single instant, and if no escape is provided, is very likely to burst the pipe. The intensity of the shock of water when stopped is made the means of raising a portion of it above the level of the head. The instrument for effecting this is called the *hydraulic ram*. At the lower end of a long pipe, P (Fig. 148), is a valve, V, opening downward;

FIG. 148.



near it, another valve, V', opens into the air-vessel, A; and from this ascends the pipe, T, in which the water is to be raised. As the valve V lies open by its weight, the water runs out, till its momentum at length shuts it, and the entire column is suddenly stopped; this impulse forces the water into the air-vessel, and thence, by the compressed air, up the tube T. As soon as the momentum is expended, the valve V drops, and the process is repeated.

214. Water-Wheels with a Horizontal Axis.—The *overshot wheel* (Fig. 149) is constructed with buckets on the circumference, which receive the water just after passing the highest point, and empty themselves before reaching the bottom. The weight of the water, as it is all on one side of a vertical diameter, causes the wheel to revolve. It is usually made as large as the fall will allow, and will carry machinery with a very small supply of water, if the fall is only considerable. The *moment* of each bucket-full constantly increases from a, where it is filled, to F, where its acting distance is radius, and therefore a maximum. From F downward the moment decreases, both by loss of water and diminution of acting distance, and becomes zero at L. These wheels deliver from 70 to 80 per cent. of the horse-power of the fall of water received upon them.

FIG. 149.

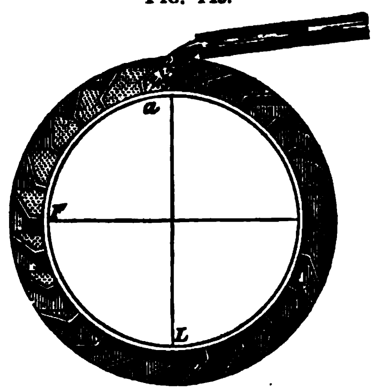
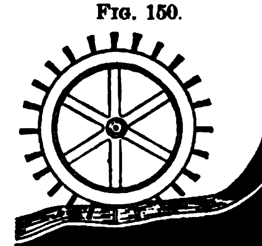


FIG. 150.

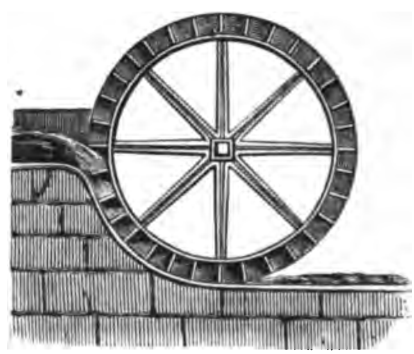


The *undershot wheel* (Fig. 150) is revolved by the momentum of running water, which strikes the float-boards on the lower side. When these are placed, as in the figure, perpendicular to the circumference, the wheel may turn either way; this is the construction adopted in tide-mills. When the wheel is required to turn only in one direction, an advantage is gained by placing the float-boards so as to present an acute angle toward the current, by which means the water acts partly by its weight, as in the overshot wheel. The undershot wheel is adapted to situations where the supply of water is always abundant.

The maximum efficiency of these wheels is obtained when the circumferential velocity is one half the velocity of the water, and is about 30 per cent. of the theoretical work of the water used. With curved float-boards the efficiency may reach about 60 per cent.

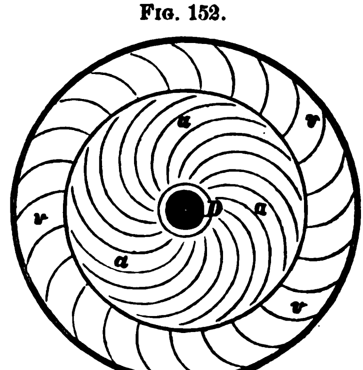
In the *breast wheel* (Fig. 151) the water is received upon the float-boards at about the height of the axis, and acts partly by its weight, and partly by its momentum. The planes of the float-boards are set at right angles to the circumference of the wheel, and are brought so near the mill-course that the water is held and acts by its weight, as in buckets. The efficiency is about 40 to 50 per cent.

FIG. 151.



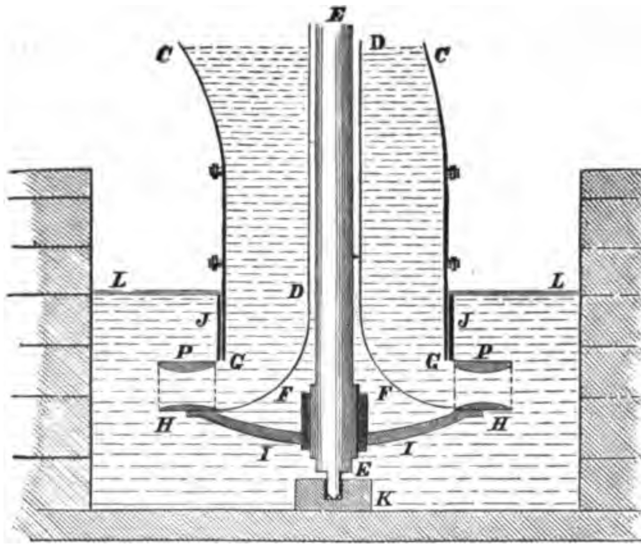
215. The Turbine.—This very efficient water-wheel, frequently called the French turbine, is of modern invention, and has received its chief improvements in this country. It revolves on a vertical axis, and surrounds the bottom of the reservoir from which it receives the water. The lower part of the reservoir is divided into a large number of sluices by curved partitions, which direct the water nearly into the line of a tangent, as it issues upon the wheel. The vanes of the wheel are curved in the opposite direction, so as to receive the force of the issuing streams at right angles. The horizontal section (Fig. 152) shows the lower part of the reservoir with its curved guides, *a*, *a*, *a*, and the wheel with its curved vanes, *v*, *v*, *v*, surrounding the reservoir; *D* is the central tube, through which the axis of the wheel passes. Fig. 153 is a vertical section of the turbine; but it does not present the guides of the reservoir, nor the vanes of the wheel. *C G*, *C G*, is the outer wall of the reservoir; *D*, *D*, its inner wall or tube; *F*, *F*, the base, curved so as to turn the

FIG. 152.



descending water gradually into a horizontal direction. The outer wall, which terminates at G, G, is connected with the base

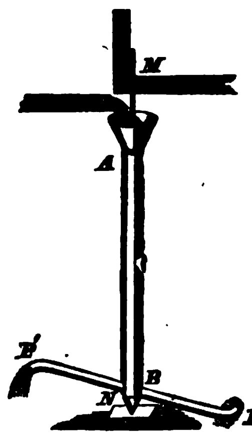
FIG. 153.



and tube by the guides which are shown at a, a, in Fig. 152. The lower rim of the wheel, H, H, is connected with the upper rim, P, P, by the vanes between them, v, v (Fig. 152), and to the axis, E, E, by the spokes I, I. The gate, J, J, is a thin cylinder which is raised or lowered between the wheel and the sluices of the reservoir. The bottom of the axis revolves in the socket K, and the top connects with the machinery. As the reservoir cannot be supported from below, it is suspended by flanges on the masonry of the wheel-pit, or on pillars outside of the wheel. To prevent confusion in the figure, the supports of the reservoir and the machinery for raising the gate are omitted. By the curved base and guides of the reservoir, the water is conducted in a spiral course to the wheel, with no sudden change of direction, and thus loses very little of its force. The wheel usually runs below the level of the water in the wheel-pit, as represented in the figure, L L being the surface of the water. The reservoir is sometimes merely the extremity of a large tapering tube or supply pipe, bent from a horizontal to a vertical direction. In such a case, the tube D D, in which the axis runs, passes through the upper side of the supply pipe. The figure represents only the lower part. The efficiency is about 80 per cent., though many claim a much higher efficiency than this.

216. Barker's Mill.—This machine operates on the principle of *unbalanced hydrostatic pressure.* It consists of a vertical hollow cylinder, A B (Fig. 154), free to revolve on its axis M N, and having a horizontal tube connected with it at the bottom. Near each end of the horizontal tube, at P and P', is an orifice, one on one side, and one on the opposite. The cylinder, being kept full of water, whirls in a direction opposite to that of the discharging streams from P and P'. This is owing to the fact that hydrostatic pressure is removed from the apertures, while on the interior of the tube, at points exactly opposite to them, are pressures which are now unbalanced, but which would be counteracted by the pressures at the apertures, if they were closed. The tube P P' may revolve either in the air, or beneath the surface of the water. The speed of rotation is increased by lengthening the tube A B.

FIG. 154.



217. Resistance to Motion in a Liquid.—The resistance which a body encounters in moving through any fluid arises from the inertia of the particles of the fluid, their want of perfect mobility among each other, and friction. Only the first of these admits of theoretical determination. So far as the inertia of the fluid is concerned, the *resistance* which a surface meets with in moving perpendicularly through it *varies as the square of the velocity.* For the resistance is measured by the *momentum* imparted by the moving body to the fluid. And this momentum (k) varies as the product of the quantity of fluid set in motion (m), and its velocity (v); or $k \propto m\,v$. But it is obvious that the quantity displaced varies as the velocity of the body, or $m \propto v$; hence $k \propto v^2$. Therefore the resistance varies as the square of the velocity.

This proposition is found to hold good in practice, where the velocity is small, as the motions of boats or ships in water; but when the velocity becomes very great, as that of a cannon ball, the resistance increases in a much higher ratio than as the square of the velocity. Since action and reaction are equal, it makes no difference, in the foregoing proposition, whether we consider the body in motion and the fluid at rest, or the fluid in motion and striking against the body at rest.

Since the resistance increases so rapidly, there is a wasteful

expenditure of force in trying to attain great velocities in navigation.

When the resistance becomes equal to the moving force, the body moves uniformly, and is said to be in a state of *dynamical equilibrium*. Thus, a body falling freely through the air by gravity does not continue to be accelerated beyond a certain limit, but is finally brought, by the resistance of the air, to a uniform motion.

218. Waves.—If a pebble be tossed upon still water, it crowds aside the particles beneath it, and raises them above the level, forming a wave around it in the shape of a ring. As soon as this ring begins to descend, it elevates above the level another portion around itself, and thus the ring-wave continues to spread outward every way from the centre. But in the meantime the water at the centre, as it rises toward the level, acquires a momentum which lifts it above that level. From that position it descends, and once more passes below the level, thus starting a new wave around it, as at first, only of less height. Hence, we see as the result of the first disturbance a series of concentric waves continually spreading outward and diminishing in height at greater distances, until they cease to be visible. In Fig. 155 (*a*) are represented three circular waves at one of the moments of time when the centre is lowest. The shaded parts are the basins or *troughs*, and the light parts, *c*, *c*, *c*, are the ridges or *crests*. Fig. 155 (*b*) is a vertical section along the line, *c*, *c*, through the centre of the system, corresponding to the momentary arrangement of (*a*). The wave centre is at *b*, and the crests at *c*, *c*, *c*. A little later, when either crest has moved half way to the place of the next one, both figures will have become reversed: the centre will be a hillock, the troughs will be at *c*, *c*, and the crests at the middle points between them.

FIG. 155 (*a*).

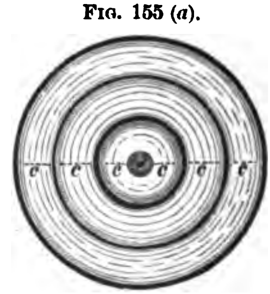
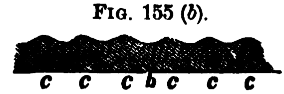


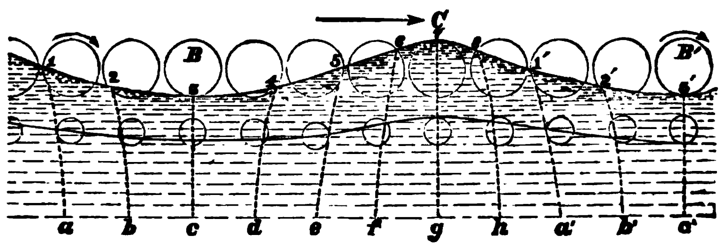
FIG. 155 (*b*).



219. Molecular Movements.—The water which constitutes a system of waves does not advance along the surface, as the waves themselves do; for a floating body is not borne along by them, but alternately rises and falls as the waves pass under it. Each particle of water, instead of advancing with the wave, describes a circular path. Within short distances from the centre of disturbance

the sizes of the circles described by the surface particles are equal, and a line connecting their centres would be parallel to the surface of the still water. Upon exciting the waves, the particles near the centre are first disturbed and commence their journey around the circle. The neighboring particles are then disturbed, and, after them, their neighbors. All the particles make a single revolution in the same time. Particles at different distances from the wave centre are, at a given instant, in different positions on their respective circles. Fig. 156 is supposed to be a vertical section through a body of water under wave disturbance. The wave

FIG. 156.



is progressing in the direction of the straight arrow. The row of circles represents the paths of a *few surface* particles. The direction of rotation of the particles is represented by the curved arrow. The wave is to be regarded as already started. Particle 1 is in the same position as it would be were there no disturbance, and is just about to commence its rotation. Particle 2 is 45° behind it in its rotation, owing to its not having been agitated as soon as 1. Likewise 3, 4, 5, 6, 7, and 8 are 90°, 135°, 180°, 225°, 270°, and 315° respectively behind 1. No. 3 is at the lowest point of its path, and is hence at the centre of the trough, while 7 is at the crest.

Particles below the surface, as far as the wave disturbance reaches, perform oscillations synchronous with those on the surface. Their paths, however, are smaller circles, or, more properly, ellipses with their longer axes in the direction of the wave propagation.

220. Phases.—Wave Length.—Whenever two particles, under the influence of wave disturbance, are at exactly the same points in their respective paths, they are said to be in *like phase.* Particles 1 and 1′, 2 and 2′ (Fig. 156) are in the like phase. Both 1 and 1′ are just about to commence their motions from their positions of equilibrium. Particles which are at diametrically opposite points of their respective paths are said to be in *opposite phase.* Particles 7 and 3 are in opposite phases. The highest points of

the crests of two waves are in like phase; the highest point of the crest and the lowest point of the trough are in opposite phase.

The length of the straight line connecting two particles in like phase is termed the *wave length.*

221. Water Wave Curve.—The sectional form of these waves is that of the inverted *trochoid*, a curve described by a point in a circle as it rolls on a straight line. The curvature of the crest is always greater than that of the trough, and the summit may possibly be a sharp ridge, in which case the section of the trough is a *cycloid*, the describing point of the rolling circle being on the circumference; the height of such waves is to their length as the diameter of a circle to the circumference. If waves are ever higher than about one-third of their length, the summits are broken into spray.

222. Velocity of Propagation.—Time of Oscillation.—During the time that a wave system propagates itself through the distance of one wave length, a particle, which happened at first to be at a crest, rotates through the lowest part of the trough around to the crest of the succeeding wave; during this time it has made a complete revolution. The time (T) necessary to make a single revolution is termed the *time* or *period of oscillation.* During this time the wave has progressed through the wave length (L). If we represent the velocity of the propagation by c, we have

$$c = \frac{L}{T}, \; L = c\,T \text{ and } T = \frac{L}{c}.$$

As in most wave motions T is a very small number, it is more convenient to employ the number of oscillations made in one second (n). Evidently

$$T = \frac{1}{n}, \; c = n\,L, \; L = \frac{c}{n} \text{ and } n = \frac{c}{L}.$$

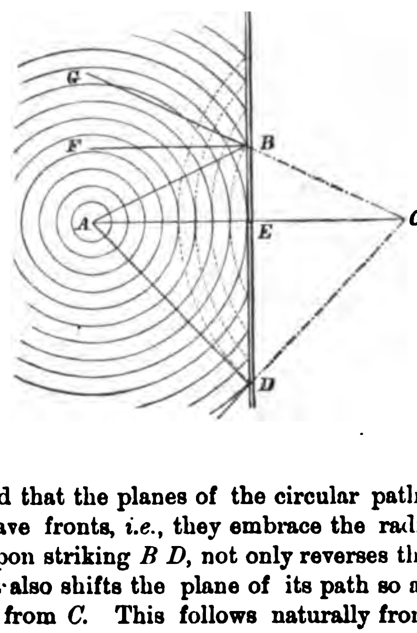
Weber says that the velocities of propagation in fluids of different specific gravity (water and quicksilver) are very nearly equal. The velocity in unconfined waters, however, is much greater than in confined. In the Atlantic Ocean the velocity is about 13 meters, while in narrow channels it is about 0.75 meter.

223. Interference of Waves.—If two systems of waves be simultaneously excited, the waves from each centre will be propagated with the same velocity as though that system alone were acting. An affected particle will describe a path, which is the result-

ant of the two paths which it would describe under the influence of each system alone. Thus a particle, which would be on a crest owing to one system and on a crest owing to the other, would still be on a crest, but of greater height than it would be from either one alone. The reverse would be the result of two combined troughs. A combination of crest and equal trough would leave the particle at rest. The results of this coexistence of wave motions are termed *interferences*.

224. Reflection of Waves.—If, in the spreading out of a wave, a particle of the wave medium, in endeavoring to describe its circular path, strike against a fixed barrier, it will, because of its elasticity, rebound and move circularly in an opposite direction. The ultimate result is that a new set of waves, moving in an opposite direction, is set up, and they are said to be *reflected* from the barrier. Let waves coming from the centre A (Fig. 157) meet the barrier $B\ D$. The dotted arcs represent the reflected waves, and they appear to come from a centre, C, such that $E\ C = A\ E$, with $A\ C$ perpendicular to $B\ D$.

FIG. 157.



It must be borne in mind that the planes of the circular paths are perpendicular to the wave fronts, *i.e.*, they embrace the radii from A. A particle at B, upon striking $B\ D$, not only reverses the direction of its rotation, but also shifts the plane of its path so as to embrace the radius $G\ C$ from C. This follows naturally from the law of elasticity (Art. 98).

225. Sea-Waves.—The waves raised by the wind rarely exhibit the precise forms above described, and the particles rarely revolve in exact circles, partly because there is scarcely ever a system of waves undisturbed by other systems, which are passing over the water at the same time, and partly because the wind, which was the original cause of the waves, acts continually upon their surfaces to distort and confuse them.

PART III.

PNEUMATICS.

CHAPTER I.

PROPERTIES OF GASES.—INSTRUMENTS FOR INVESTIGATION.

226. Gases Distinguished from Liquids.—The property of *mobility* of particles, which belongs to all fluids, is more remarkable in gases than in liquids.

While gaseous substances are compressed with ease, they are always ready to expand and occupy more space. This property, called *dilatability,* scarcely belongs to liquids at all.

This property may be experimentally illustrated by placing a bag only partly full of air under the receiver of an air pump and exhausting the air; the external pressure having been removed, the bag will seem full almost to bursting, the contained air having dilated to many times its former volume.

Invert a flask containing air into a beaker of colored water, and place the whole under the receiver of an air pump. As the air is exhausted the contained air in the flask will expand and, driving the water out of the neck of the flask, will rise in bubbles to the surface. Upon admitting air again to the receiver, the water will be forced into the flask to take the place of the escaped air, and will rise until the tension of the contained air, together with the weight of the water column, is equal to that of the air in the receiver.

227. Tension of Gases.—By the term *tension* just used, is meant the force exerted by the gas at each instant in opposition to any compressing or restraining force; or, in other words, the force of expansion. The molecules of the gas are supposed to be flying through space with great velocity in straight lines. The combined effect of the impact of these molecules upon the walls of the containing vessel is an outward pressure which is opposed by the strength of the material of the vessel. In the first experi-

ment given, the impact of the molecules of the air in the room upon the *outside* of the bag counterbalanced the impact of the molecules of the contained air upon the *inside;* but when the external air was removed, there was no counterbalancing force, until the bag expanded so much that the strength of elasticity of the rubber itself equaled the resultant of the impacts within. This theory of tension will be of great help in discussing the subject of expansion by heat.

228. Change of Condition.—Liquids, and even solids, may be changed into the gaseous or aeriform condition by heating them sufficiently. By being cooled, they return again to their former state. In the gaseous form they are called *vapors.* All substances which are ordinarily gases can be so far cooled, especially under great pressure, as to be reduced to the liquid or solid form.

Those which can only be thus reduced under very great pressures, and at very low temperatures, are regarded as types of a theoretically perfect gas.

229. Diffusion of Gases.—If two flasks, *A* and *B*, be connected by a tube, as in Fig. 158, and the upper, *A*, be filled with hydrogen, or illuminating gas, and the lower *B* with carbonic dioxide, after a time some of the lighter gas will be found in *B*, having passed down through the tube, while a part of the heavy gas in *B* will have passed upwards to *A*. This result must follow from the theory of molecular motion given before. The action is called *diffusion.*

FIG. 158.

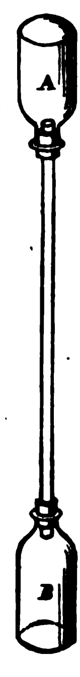


FIG. 159.

230. Osmose of Gases.—Cement a glass tube, about twenty-four inches long, to a porous cell *B* (Fig. 159), and dip the lower end of the tube into colored liquid *A*. Now fill an inverted bell jar with hydrogen, or illuminating gas, and place it over *B*. Either will make its way into the porous cell more rapidly than the air makes its way out, diffusion inwards being more rapid than diffusion outwards, and, in consequence, some air will be driven out through the glass tube, escaping in bubbles through the liquid in *A*. Upon removing the bell jar the gas within the porous cell will pass out again more rapidly than air can pass in, and a partial vacuum will be

formed, causing a rise of the colored liquid in the tube. This mixing of gases through a porous cell, or a thin moistened membrane, is denominated *Osmose of Gases.*

Under equal pressures, the densities of gases are inversely as the squares of the velocities, with which equal volumes will pass through the same narrow opening.

231. Weight of Gases.—Like all other forms of matter, gases have weight. Some are relatively light, some heavy. Take a copper globe, and hang it upon one scale pan of a delicate balance, and accurately counterpoise it. Next exhaust the air from the globe, and it will be found lighter than before; fill with carbonic dioxide, and it will weigh much more than at first. The heavier gases may be poured from one vessel to another like water; carbonic dioxide may be poured from a beaker upon a burning candle, which may thus be extinguished.

232.—Pressure of Gases.—As a consequence of the weight of gases we have to consider the pressure exerted by them. We shall use the atmosphere as a type of all gases. Across the open top of a cylindric receiver stretch a sheet of rubber; upon exhausting the air from the receiver, the rubber will be pressed inwards by the external air. Substitute for the sheet of rubber a sheet of wetted bladder, which allow to dry. Upon exhausting the air the bladder will burst, under the pressure inwards, with a loud report.

Exhaust the air from a receiver, into which projects a jet tube closed with a stop-cock; upon submerging the outer end of the jet and opening the stop-cock, a fountain in vacuo will be produced.

Exhaust the air from two closely fitted hemispheres, called Magdeburg hemispheres, of about four inches diameter; a force of over 175 lbs. will be required to separate them.

Having a cylinder about five inches in diameter, with closely fitted piston, attach a weight of 250 lbs. to the lower side of the piston, exhaust the air from the cylinder above the piston, and the weight will be raised.

The pressure of the air upon our bodies and the outward pressure of the blood against the walls of the small veins and capillaries are in equilibrium. Place the palm of the hand upon the broad opening of a receiver, called a hand glass, and exhaust the air beneath; the air pressure being removed, the flesh will protrude into the receiver, and the skin, by its redness, will give evidence of the engorgement of the blood-vessels.

233. Buoyancy.—When the water displaced by an immersed

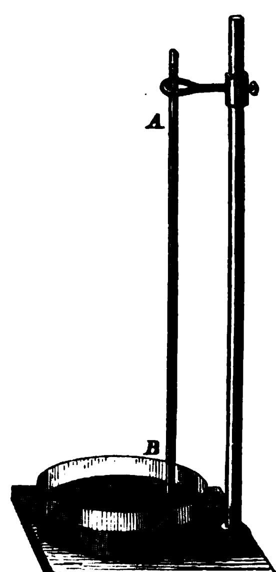
body weighs more than the body itself, it will rise to the surface and float; so too when the volume of air displaced by a body weighs more than that body, it will rise and float in the air. Attach the gas jet to a clay pipe by rubber tubing, and blow soap-bubbles; these will rise rapidly to the ceiling of the room. Blow a small bubble, and then transfer the end of the tube to the gas jet and enlarge the bubble till its specific gravity is about the same as that of the air; it will now float about the room, sometimes rising, sometimes falling, until it bursts.

Large balloons have ascended to a height of seven miles.

If a large and a small body are in equilibrium on the two arms of a balance, and the whole be set under a receiver, and the air be removed, the larger body will preponderate, showing that it is really the heaviest. Their apparent equality of weight when in the air is owing to its buoyant power, which diminishes the apparent weight of an immersed body by just the weight of the displaced fluid. Hence, the larger the body, the more weight it loses.

If the air be exhausted from a tube, four or five feet long and two inches in diameter, containing a small coin and a feather, it will be found, upon quickly inverting the tube, that the coin and the feather will fall through its length in the same time, both having the same velocity; if it were not for the obstruction of the air, all bodies would fall to the earth with the same velocity.

FIG. 160.



234. Torricelli's Experiment.—A glass tube *A B* (Fig. 160) about three feet long, and hermetically sealed at one end, is filled with mercury, and then, while the finger is held tightly on the open end, it is inverted in a cup of mercury. On removing the finger after the end of the tube is beneath the surface of the mercury, the column sinks a little way from the top, and there remains. Its height is found to be nearly thirty inches above the level of mercury in the cup. If sufficient care is taken to expel globules of air from the liquid, the space above the column in the tube is as perfect a vacuum as can be obtained. It is called the *Torricellian* vacuum, from Torricelli of Italy, a

disciple of Galileo, who, by this experiment, disproved the doctrine that *nature abhors a vacuum*, and fixed the limits of atmospheric pressure.

235. Pressure of Air Measured.—The column is sustained in the Torricellian tube by the pressure of air on the surface of mercury in the vessel; for the level of a fluid surface cannot be preserved unless there is an equal pressure on every part. Hence, the column of mercury on one part, and the column of air on every other equal part, must press equally. To determine, therefore, the pressure of air, we have only to weigh the column of mercury, and measure the area of the mouth of the tube. If this is carefully done, it is found that the weight of mercury is about 14.7 lbs. on a square inch. Therefore the atmosphere presses on the earth with a force of nearly 15 pounds to every square inch, or more than 2000 lbs. per square foot.

The specific gravity of mercury is about 13.6; and therefore the height of a column of water in a Torricellian tube should be 13.6 times greater than that of mercury, that is, about 34 feet. Experiment shows this to be true. And it was this significant fact, that *equal weights* of water and mercury are sustained in these circumstances, which led Torricelli to attribute the effect to a common force, namely, the pressure of the air.

236. Pascal's Experiment.—As soon as Torricelli's discovery was known, Pascal of France proposed to test the correctness of his conclusion, by carrying the apparatus to the top of a mountain, in order to see if less air above the instrument sustained the mercury at a less height. This was found to be true; the column gradually fell, as greater heights were attained. The experiment of Pascal also determined the relative density of mercury and air. For the mercury falls one-tenth of an inch in ascending 87.2 feet; therefore the weight of the one-tenth of an inch of mercury was balanced by the weight of the 87.2 feet of air. Therefore the specific gravities of mercury and air (being inversely as the heights of columns in equilibrium) are as $(87.2 \times 12 \times 10 =)$ 10464 : 1. In the same way it is ascertained that water is 770 times as dense as air. These results can of course be confirmed by directly weighing the several fluids, which could not be done before the invention of the air-pump.

That it is the atmospheric pressure which sustains the column of mercury may be shown thus: Place the Torricellian tube and cistern under a receiver, made for the purpose, and exhaust the air; the mercury will fall lower and lower at each stroke of the pump, until, if the pump be in good working order, the column will be nearly at the level of the mercury in the cistern.

237. Mariotte's Law.—

At a given temperature, the volume of air is inversely as the compressing force.

An instrument constructed for showing this is called *Mariotte's tube*. The end B (Fig. 161) is sealed, and A open. Pour in small quantities of mercury, inclining the tube so as to let air in or out, till both branches are filled to the zero point. The air in the short branch now has the same tension as the external air, since they just balance each other. If mercury be poured in till the column in the short tube rises to C, the inclosed air is reduced to one-half of its original volume, and the column A in the long branch is found to be 29 or 30 inches above the level of C, according to the barometer at the time. Thus, *two* atmospheres, one of mercury, the other of air above it, have compressed the inclosed air into *one-half* its volume. If the tube is of sufficient length, let mercury be poured in again, till the air is compressed to one-third of its original space; the long column, measured from the level of the mercury in the short one, is now twice as high as before; that is, *three* atmospheres, two of mercury and one of air, have reduced the same quantity of air to *one-third* of its first volume. This law has been found to hold good in regard to atmospheric air up to a pressure of nearly thirty atmospheres.

FIG. 161.

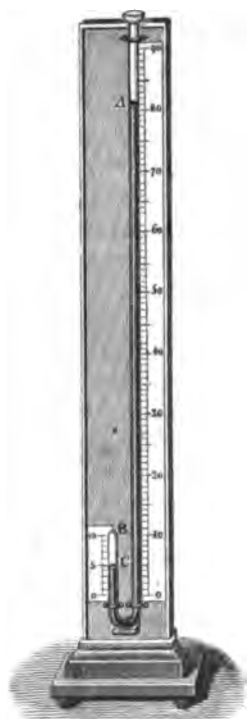
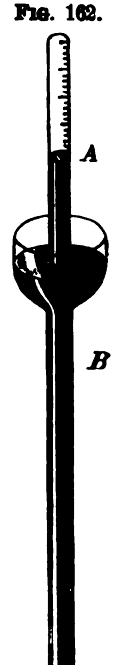


FIG. 162.



On the other hand, if the pressure on a given mass of air is diminished, its volume is found to increase according to the same law. When the pressure is *half* an atmosphere, the volume is *doubled;* when *one-third* of an atmosphere, the volume is three times as great, &c.

This may be shown by filling a tube A, closed at one end, nearly full of mercury, and inverting it in a tube-like cistern B, as in Fig. 162. Suppose that the tube, of uniform bore, contains an inch of air when the mercury is at

the same level within and without; upon raising the tube until the column within is about fifteen inches high the air will be found to occupy two inches of space. At the beginning of the experiment the mercury being at the same level in both tubes, the pressure upon the contained air was due to the atmosphere, and was about 14.7 lbs. per square inch. At the close the pressure was less by the force necessary to sustain the fifteen inches of mercury, leaving only a pressure of about one-half an atmosphere. Experiments in this case will not be satisfactory unless precautions be taken to remove the air bubbles which adhere to the glass tube. Since the tension of the inclosed air always balances the compressing force, and since the density is inversely as the volume, it follows from Mariotte's Law that, when the temperature is the same,

The tension of air varies as the compressing force; and *The tension of air varies as its density.*

This law is, however, not strictly true of gases, except when far removed from the critical point; when near the point of condensation the departure from the law is most marked. Even in the ordinary state the law is not *strictly* followed by many gases.

238. Dalton's Law.—

At a given temperature the tension of a mixture of gases is equal to the sum of the tensions of the gases taken separately.

In this law a *mixture* is spoken of and not a chemical combination. Each gas diffuses and is found equally distributed throughout the containing vessel just as though no other gas were present, differing in this respect from mechanical mixtures of liquids, such as oil and water, in which the components of the mixture arrange themselves according to their specific gravities. If a vessel *A* contains a cubic foot of nitrogen at a tension of ten lbs. per square inch, and a perfect vacuum *B* of the same capacity be connected with this, and *all the gas* be transferred to the new vessel *B*, the tension in the latter case will be the same as in the former, ten lbs. Now, if a cubic foot of oxygen at a tension of ten lbs. be transferred to the vessel *B* also, it will exert a pressure of ten lbs. just as though the nitrogen were not present, giving a total pressure of twenty lbs. for the mixture. These two illustrations have been given to prevent any misapprehension which may arise from the following frequently repeated and very concise wording of the law: "Every gas acts as a vacuum with respect to every other."

239. Laws of Mixture of Gases and Liquids.—Water and many other liquids will contain gases in solution; but under the same conditions of temperature and pressure, a given liquid

does not absorb equal quantities of different gases. For example, at the mean temperature and pressure water dissolves about .025 of its volume of nitrogen, .046 of its volume of oxygen, its own volume of carbonic dioxide, and 430 times its volume of ammonia. Mercury, on the other hand, does not dissolve any of the gases.

Experiment has determined the three following laws of the mixture of liquids and gases.

1st. *For the same gas, the same liquid, at a constant temperature, the weight of gas absorbed is proportional to the pressure.* From this it follows that the volume dissolved is constant, whatever may be the changes in pressure, or, what is the same thing, the density of the gas absorbed bears a constant ratio to that of the gas not absorbed.

2d. *The quantity of gas absorbed increases as the temperature decreases.*

3d. *The quantity of a gas which a given liquid will dissolve is independent of the kinds and quantities of other gases which may already be held in solution.*

If instead of a single gas in contact with the liquid, a mixture of several gases be used, each of these will be dissolved in the quantity due to its proportional part of the total pressure. For example, since oxygen forms only about one-fifth the volume of the air, water under ordinary conditions absorbs the same quantity of oxygen, as if the atmosphere were wholly of oxygen under a pressure of $\frac{14.7}{5}$ lbs.

The first law may be experimentally illustrated by opening a bottle of common soda water. As soon as the cork is loosened it is driven out by the tension of the confined carbonic dioxide above the liquid, and the pressure being reduced by the escape of the free gas, the absorbed gas is at once given off in bubbles, the escape of which produces foaming.

If after all the gas has seemingly escaped, a portion of the liquid be poured into a beaker and placed under the receiver of the air pump, a fresh discharge of bubbles will follow the first stroke of the pump, consequent upon the still further reduction of pressure.

A portion poured into a flask and heated will serve to illustrate the second law, the rise in temperature causing a constant rise of bubbles of gas.

240. The Barometer.—When the Torricellian tube and basin are mounted in a case, and furnished with a graduated scale, the instrument is called a barometer. The scale is divided into

millimeters or inches and tenths, and usually extends for a space more than sufficient to include all the natural variations in the weight of the atmosphere. By attaching a vernier to the scale, as is commonly done in meteorological observations, the reading may be carried to the fractional parts of the scale. By observing the barometer from day to day, and from hour to hour, it is found that the atmospheric pressure is constantly fluctuating.

As the meteorological changes of the barometer are all comprehended within a range of two or three inches, much labor has been expended in devising methods for magnifying the motions of the mercurial column, so that more delicate changes of atmospheric pressure might be noted. The inclined tube and the wheel barometer are intended for this purpose. A description of these contrivances, however, is unnecessary, as they are all found to be inferior in accuracy to the simple tube and basin.

241. Corrections for the Barometer.—

1. For *change of level in the basin.*—The numbers on the barometer scale are measured from a certain zero point, which is assumed to be the level of the mercury in the basin. If now the column falls, it raises the surface in the basin; and if it rises, it lowers it. If the basin is broad, the change of level is small, but it always requires a correction. To avoid this source of error, the bottom of the basin is made of flexible leather, with a screw underneath it, by which the mercury may be raised or lowered, till its surface touches an index that marks the zero point. This adjustment should always be made before reading the barometer.

2. For *capillarity.*—In a glass tube mercury is depressed by capillary action (Art. 200). The amount of depression is less as the tube is larger. This error is to be corrected by the manufacturer, the scale being put below the true height by a quantity equal to the depression.

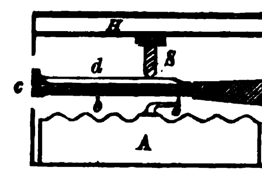
There is a slight variation in this capillary error, arising from the fact that the rounded summit of the column, called the *meniscus*, is more convex when ascending than when descending. To render the meniscus constant in its form, the barometer should be jarred before each reading.

3. For *temperature.*—As mercury is expanded by heat and contracted by cold, a given atmospheric pressure will raise the column too high, or not high enough, according to the temperature of the mercury. A thermometer is therefore attached to the barometer, to show the temperature of the instrument. By a table of corrections, each reading is reduced to the height the mercury would have if its temperature was 0° C.

4. For *altitude of station.*—Before comparing the observations of different places, a correction must be made for altitude of station, because the column is shorter according as the place is higher above the sea level.

242. The Aneroid Barometer.—This is a small and portable instrument, in appearance a little like a large chronometer. The essential part of this barometer is a flat cylindrical metallic box, shown in section at A (Fig. 163), whose upper surface is corrugated, so as to be yielding.

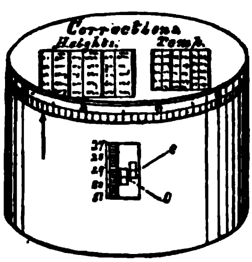
Fig. 163.



The box being partly exhausted of air, the external pressure causes the top to sink in to a certain extent; if the pressure increases, the surface descends a little more; if it diminishes, a little less.

These small movements are communicated to a lever b (Fig. 163) whose end c moves over a scale of inches on the case (Fig. 164). To insure contact of the pin o (Fig. 163), and also to secure a constant resistance to the motion of the lever b a spring d is attached to the lever b near the fulcrum, which is pressed upon by the screw s, whose graduated head H is of the same diameter as the box. The end of this spring e (Fig. 164) must be brought to coincide with the end of the lever, by turning the screw-head H. The reading of inches and tenths is taken from the scale of inches, the hundredths and thousandths being given by the screw-head. This is one of the most simple of all the aneroids in construction. A table of corrections for temperature, and reductions to the standard mercurial barometer is entered upon the cover.

Fig. 164.



243. Pressure and Latitude.—The mean pressure of the atmosphere at the level of the sea is 761 mm. or 30 inches. But it is not the same at all latitudes. From the equator either northward or southward, the mean pressure increases to about latitude 30°, by 4 or 5 millimeters, and thence decreases to about 65°, where the pressure is less than at the equator, and beyond that it slightly increases. This distribution of pressures in zones is due to the great atmospheric currents, caused by heat in connection with the earth's rotation on its axis.

The amount of variation in barometric pressure is very unequal in different latitudes; and in general, the higher the latitude, the

greater the variation. Within the tropics the extreme range scarcely ever exceeds one-fourth of an inch, while at latitude 40° it is more than two inches, and in higher latitudes even reaches three inches.

244*. Diurnal Variation.—If a long series of barometric observations be made, and the mean obtained for each hour of the day, the changes caused by weather become eliminated, and the diurnal oscillation reveals itself. It is found that the pressure reaches a maximum and a minimum twice in 24 hours. The times of greatest pressure are from 9 to 11, and of least pressure from 3 to 5, both A. M. and P. M. In tropical climates this variation is very regular, though small; but in the temperate zones the irregular fluctuations of weather conceal it in a great degree.

This daily fluctuation of the barometer is caused by the changes which take place from hour to hour of the day in the temperature, and by the varying quantity of vapor in the atmosphere.

The surface of the globe is always divided into a day and night hemisphere, separated by a great circle which revolves with the sun from east to west in twenty-four hours. The hemisphere exposed to the sun is warm, the other is cold. The time of greatest heat is not at noon, when the sun is in the meridian, but about two or three hours after; the period of greatest cold occurs about four in the morning. As the hemisphere under the sun's rays becomes heated, the air, expanding upwards and outwards, flows over upon the other hemisphere where the air is colder and denser. There thus revolves round the globe from day to day a wave of heated air, from the crest of which air constantly tends to flow towards the meridian of greatest cold on the opposite side of the the globe.

The barometer is influenced to a large extent by the elastic force of the vapor of water invisibly suspended in the atmosphere, in the same way as it is influenced by the dry air (oxygen and nitrogen). But the vapor of water also exerts a pressure on the barometer in another way. Vapor tends to diffuse itself equally through the air; but as the particles of air offer an obstruction to the watery particles, about 9 or 10 A. M., when evaporation is most rapid, the vapor is accumulated or pent up in the lower stratum of the atmosphere, and being impeded in its ascent its elastic force is increased by the reaction, and the barometer consequently rises. When the air falls below the temperature of the dew-point part of its moisture is deposited in dew, and since some time must elapse before the vapor of the upper strata can diffuse itself downwards to supply the deficiency, the barometer falls,—

most markedly at 10 P. M., when the deposition of dew is greatest.

245. The Barometer and the Weather.—The changes in the height of the barometer column depend directly on nothing else than the atmospheric pressure. But these changes of pressure are due to several causes, such as wind and changes of temperature and moisture.

The practice formerly prevailed of engraving at different points of the barometer scale several words expressive of states of weather, "fair, rain, frost, wind," &c. But such indications are worthless, being as often false as true; this is evident from the fact that the height of the column would be changed from one kind of weather to another by simply carrying the instrument to a higher or lower station.

No general system of rules can be given for anticipating changes of weather by the barometer, which would be applicable in different countries. Rules found in English books are of very little value in America.

Severe and extensive storms are almost always accompanied by a fall of the barometer while passing, and succeeded by a rise of the barometer.

246. Heights Measured by the Barometer.—Since mercury is 10464 times as heavy as air (Art. 236), if the barometer is carried up until the mercury falls one inch, it might be inferred that the ascent is 10464 inches, or 872 feet. This would be the case if the density were the same at all altitudes. But, on account of diminished pressure, the air is more and more expanded at greater heights. Besides this, the height due to a given fall of the mercury varies for many reasons, such as the temperature of the air, the temperature of the mercury, the elevation of the stations, and their latitude. Hence, the measurement of heights by the barometer is somewhat troublesome, and not always to be relied on. Formulæ and tables for this purpose are to be found in practical works on physics.

247. The Gauge of the Air-Pump.—The Torricellian tube is employed in different ways as a gauge for the air-pump, to indicate the degree of exhaustion. In Fig. 166 the gauge *G* is a tube about 33 inches long, both ends of which are open, the lower immersed in a cup of mercury, and the upper communicating with the interior of the receiver. As the exhaustion proceeds, the pressure is diminished within the tube, and the external air raises the mercury in it. A perfect vacuum would be indicated by a height of mercury equal to that of the barometer at the time.

Another kind of gauge is a barometer already filled, the basin of which is open to the receiver. As the tension of air in the receiver is diminished, the column descends, and would stand at the same level in both tube and basin, if the vacuum were perfect.

FIG. 165.

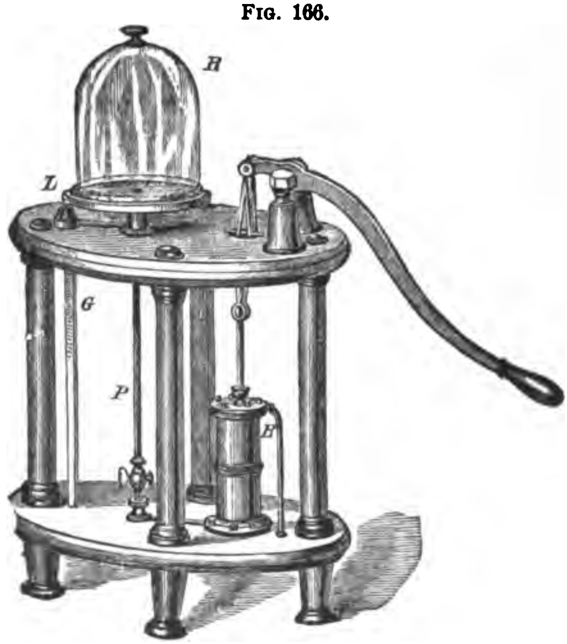
A modified form of the last, called the *siphon gauge*, is the best for measuring the rarity of the air in the receiver when the vacuum is nearly perfect. Its construction is shown by Fig. 165. The top of the column, A, is only 5 or 6 inches above the level of B in the other branch of the recurved tube. As the air is withdrawn from the open end C, the tension at length becomes too feeble to sustain the column; it then begins to descend, and the mercury in the two branches approaches a common level.

CHAPTER II.

INSTRUMENTS WHOSE OPERATION DEPENDS ON THE PROPERTIES OF AIR.

248. The Air-Pump.—This is an instrument by which nearly all the air can be removed from a vessel or receiver. It has a variety of forms, one of which is shown in Fig. 166. In the

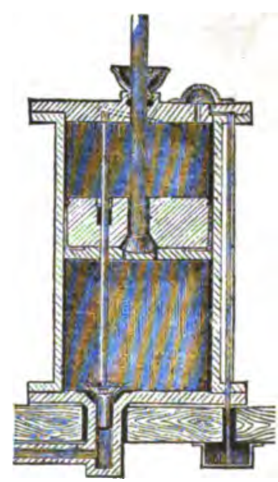
FIG. 166.



barrel B an air-tight piston is alternately raised and depressed by the lever, the piston-rod being kept vertical by means of a guide. The pipe P connects the bottom of the barrel with the brass plate L, on which rests the receiver R. The surface of the plate and the edge of the receiver are both ground to a plane. G is the gauge which indicates the degree of exhaustion. There are three valves, the first at the bottom of the barrel, the second in the piston, and the third at the top of the barrel. These all open upward, allowing the air to pass out, but preventing its return.

249. Operation.—When the piston is depressed, the air below it, by its increased tension, presses down the first valve, and opens the second, and escapes into the upper part of the barrel. When the piston is raised, the air above it cannot return, but is pressed through the third valve into the open air; while the air in the receiver and pipe, by its tension, opens the first valve, and diffuses itself equally through the receiver and barrel. Another descent and ascent only repeat the same process; and thus, by a succession of strokes, the air is nearly all removed.

FIG. 167.



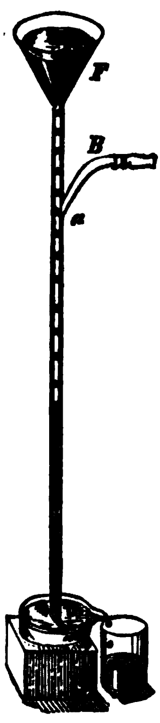
The exhaustion can be made more complete if the first and second valves are opened by the action of the piston and rod, rather than by the tension of the air. This method is illustrated by Fig. 167, a section of the barrel and piston. The first and second valves, as shown in the figure, are conical or *puppet* valves, fitting into conical sockets. The first has a long stem attached, which passes through the piston air-tight, and is pulled up by it a little way, till it is arrested by striking the top of the barrel. The second valve is a conical frustum on the end of the piston-rod. When the rod is raised, it shuts the valve before moving the piston; when it begins to descend, it opens the valve again before giving motion to the piston. The first valve is shut by a lever, which the piston strikes at the moment of its reaching the top. The oil which is likely to be pressed through the third valve is drained off by the pipe (on the right in both figures) into a cup below the pump.

250. Rate of Exhaustion.—The quantity removed, by successive strokes, and also the quantity remaining in the receiver, diminishes in the same geometrical ratio. For, of the air occupy-

ing the barrel and receiver, a barrel-full is removed at each stroke, and a receiver-full is left. If, for example, the receiver is *three* times as large as the barrel, the air occupies *four* parts before the descent of the piston; and by the first stroke *one-fourth* is removed, and *three-fourths* are left. By the next stroke, three-fourths as much will be removed as before ($\frac{1}{4}$ of $\frac{3}{4}$, instead of $\frac{1}{4}$ of the whole), and so on continually. The quantity left obviously diminishes also in the same ratio of three-fourths. In general, if b expresses the capacity of the barrel, and r that of the receiver and connecting-pipe, the ratio of each descending series is $\frac{r}{b+r}$. With a given barrel, the rate of exhaustion is obviously more rapid as the receiver is smaller. If the two were equal, ten strokes would rarefy the air more than a thousand times. For $(\frac{1}{2})^{10} = \frac{1}{1024}$.

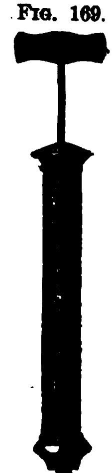
As a term of this series can never reach zero, a complete exhaustion can never be effected by the air-pump; but in the best condition of a well-made pump, it is not easy to discover by the gauge that the vacuum is not perfect.

FIG. 168.



251. Sprengel's Pump.—This apparatus is too slow in its action for ordinary lecture illustration, but gives a much better vacuum than any piston pump. The length *a b* (Fig. 168) must be more than 30 inches, and the diameter of the tube should be quite small, about $\frac{1}{16}$ inch. Mercury from the funnel *F* falls down the tube *a b* in drops, which carry air before them from the receiver, which is connected with the exhaust branch *B* by suitable tubing.

FIG. 169.



252. The Air Condenser.—While the air-pump shows the tendency of air to *dilate* indefinitely, as the compressing force is removed, another useful instrument, the *condenser*, exhibits the indefinite compressibility of air. Like the pump, it consists of a barrel and piston, but its valves, one in the piston and one at the bottom of the barrel, open downward. Fig. 169 shows the exterior of the instrument. If it be screwed upon the top of a strong receiver (Fig. 170), with a stop-cock connecting them, air may be forced in, and then secured by shutting the stop-cock.

When the piston is depressed, its own valve is shut by the increased tension of the air beneath it, and the lower one opened by the same force. When the piston is raised, the lower valve is kept shut by the condensed air in the receiver, and that of the piston is opened by the weight of the outer air, which thus gets admission below the piston.

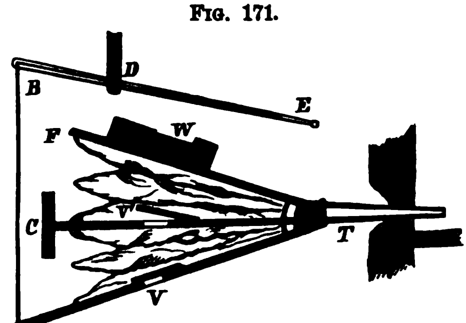
Fig. 170.

The quantity of air in the receiver increases at each stroke in an arithmetical ratio, because the same quantity, a barrel-full of common air, is added every time the piston is depressed. A small Mariotte's tube is attached to the receiver, to show how many atmospheres have been admitted.

253. Experiments with the Air Condenser.—If the receiver be partly filled with water, and a pipe from the stop-cock extend into it, then when the condenser has been used and removed, and the stop-cock opened, a jet of water will be thrown to a height corresponding to the tension of the inclosed air. A gas-bag being placed in the condenser, then filled and shut, will become flaccid when the air around it is compressed. A thin glass bottle, sealed, will be crushed by the same force. By these and other experiments may be shown the effects of increased tension.

254. The Bellows.—The simple or *hand-bellows* consists of two boards or lids hinged together, and having a flexible leather round the edges, and a tapering tube through which the air is driven out. In the lower board there is a hole with a valve lying on it, which can open inward. On separating the lids, the air by its pressure instantly lifts the valve and fills the space between them; but when they are pressed together, the valve shuts, and the air is compelled to escape through the pipe. The stream is intermittent, passing out only when pressure is applied.

Fig. 171.

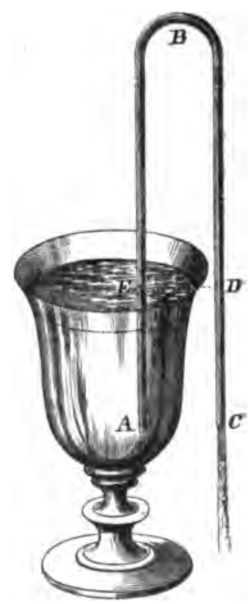


The *compound bellows*, used for forges where a constant stream is needed, are made with two compartments. The partition, $C\ T$ (Fig. 171) is fixed, and has in it a valve V' opening upward. The lower lid has also a valve V opening upward, and the upper one is loaded with weights.

The pipe T is connected with the upper compartment. As the lower lid is raised by the rod $A\ B$, which is worked by the lever $E\ B$, the air in the lower part is crowded through V' into the upper part, whence it is by the weights pressed through the pipe T in a constant stream. When the lower lid falls, the air enters the lower compartment by the valve V.

255. The Siphon.—If a bent tube $A\ B\ C$ (Fig. 172) be filled, and one end immersed in a vessel of water, the liquid will be discharged through the tube so long as the outer end is lower than the level in the vessel. Such a tube is called a *siphon*, and is much used for removing a liquid from the top of a reservoir without disturbing the lower part. The height of the bend B above the fluid level must be less than 34 feet for water, and less than 30 inches for mercury. The reasons for the motion of the water are, that the atmosphere is able to sustain a column higher than $E\ B$, and that $C\ B$ is longer than $E\ B$. The two pressures on the highest cross-section B of the tube are unequal.

FIG. 172.



For the pressure at B towards the right is equal to the atmospheric pressure, which call a, minus the weight of the column $E\ B$, which call b; or $P = a - b$. The pressure towards the left at B is equal to a minus the weight of the column $C\ B$, greater than $E\ B$, and this weight we may call $b + c$; or $P' = a - (b + c)$. The difference of these pressures will determine the motion at B.

$$P - P' = (a - b) - \{a - (b + c)\} = c,$$

and this excess of pressure c causes a flow in the direction $E\ B\ C$.

The excess of pressure at any other point of the siphon might have been discussed in the same general way. In no case will water flow if the short arm exceeds 34 feet in length, and practically it must be less than this.

If the tube is small, it may be filled by suction, after the end A is immersed. If it is large, it may be inverted and filled, and then secured by stop-cocks, till the end is beneath the water.

256. Siphon Fountain.—In order that the flow may be maintained, it is not necessary that the tube should contain noth-

ing but liquid. Air may collect in large quantities at the highest point and still not wholly stop the action.

Into a flask F (Fig. 173) fit an air-tight cork, through which pass two tubes, one b entering several inches into the flask and terminating in a fine jet a, and the other d c ending at the cork. Through the tube d c pour water till the flask is filled to the jet a, when inverted as in the figure. Place the end of the tube a b in a beaker of water H, and let the end of a rubber tube lead from d to a pail upon the floor. The water in the flask will flow out through the tube c d, and when the tension of the air in F has been sufficiently lowered, the pressure of the atmosphere upon the water in the beaker H will force it up the tube b a and out through the jet. The action will continue, as in any other form of siphon, so long as water is supplied to the short arm.

FIG. 173.

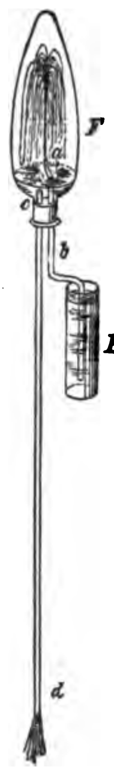
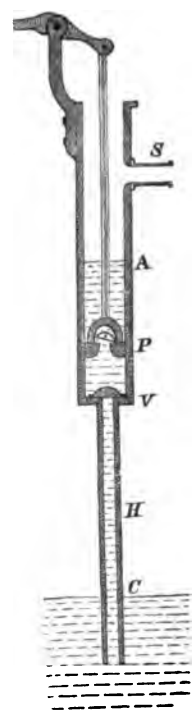


FIG. 174.



257. The Suction Pump.— The section (Fig. 174) exhibits the construction of the common suction pump. By means of a lever, the piston P is moved up and down in the tube A V. In the piston is a valve opening upward, and at the top of the pipe V C is another valve, shown at V, also opening upward. The latter valve must be at a less height than 34 feet above the water C, the practical limit being about 29 feet, depending somewhat upon the weight of the valves. When the piston P is raised, its valve is kept shut by the pressure of the atmosphere above. The air below the piston in the barrel A V is rarefied and presses less and less upon the valve V until at last its tension, together with the weight of the valve, is less than the tension of the air in the pipe V C and the valve opens, the air passing through from below. Now the tension of the air in V C being less than that of the atmosphere, a column of water will be forced up the pipe to a height such that the tension of the air in the pipe together with the weight of the column of water shall equal the pressure of the external air.

When the upward motion of P ceases, the valve V closes by its own weight. When P descends, on the return stroke, the air between it and V is compressed till its tension is greater than that of the atmosphere and the weight of the valve combined, when the valve in P is raised and the compressed air escapes. The piston being raised again, the water rises still higher, till at length it passes through the valve, and the piston dips into it; after this the water above P is lifted to the discharge spout S, while that below P is forced to follow the piston in its upward motion by the pressure of the atmosphere, as before.

258. Calculation of the Force.—To determine the force necessary to be applied to the piston-rod in pumping, let us neglect the weight of rod, piston, and valve as well as the friction.

Suppose the water to have risen to the point H (Fig. 174). Let $m =$ grams per sq. cm. downward pressure of the column $H\ C$, and $r =$ grams per sq. cm. tension of the rarified air in the tube. These two tend to drive the water downward but are balanced by the atmospheric pressure ($=$ 1033 grams per sq. cm.) in the cistern. As the water is in equilibrium we have $m + r = 1033$;

$$\therefore r = 1033 - m.$$

Now, if the area of the piston is Q sq. cm., its upper surface suffers a pressure of 1033 Q grams. Upon the under surface the pressure is ($Q\ r =$) $Q\ (1033 - m)$ grams. The difference of these gives the lifting force,

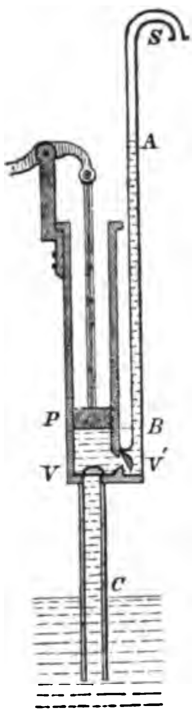
$F = [1033\ Q - (1033\ Q - m\ Q) =]\ m\ Q$ grams, that is to say, the lifting force equals the weight of a column of water whose cross-section equals the area of the piston and whose height is $H\ C$. If Q and $H\ C$ are measured in cms., their product gives the force in grams.

Suppose the water to be above the piston, at A. Call the downward pressure of the column $A\ P$, m' grams per sq. cm., and the downward pressure of the column $P\ C$, n' grams per sq. cm. The pressure upon the upper side of P is $Q\ (1033 + m')$ grams, and pressure upon the lower side is $Q\ (1033 - n')$ grams. The difference equals the lifting force,

$F = [1033\ Q + m'\ Q - 1033\ Q + n'\ Q] = Q\ (m' + n')$ grams, or is equal to the weight of a column of water of cross-section Q and height $A\ C$, as in the previous case. From this calculation we learn that only the area of the piston and height of water in the pump above the surface of the cistern need be considered, the diameter of the pipe $V\ C$ not entering the calculation.

259. The Forcing Pump.—The piston of the forcing pump (Fig. 175) is solid, and the upper valve V' opens into the side pipe V' S. In the ascent of the piston, the water is raised as in the suction pump; but in its descent, a force must be applied to press the water which is above V into the side pipe through V'.

FIG. 175.

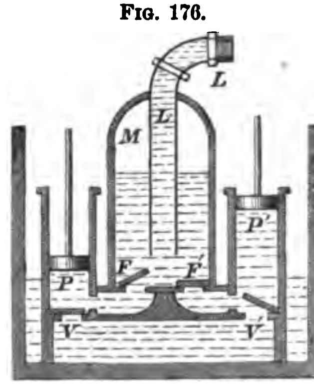


Let $P\,C = h$ cm., $B\,A = h'$ cm., and the diameter of the piston $= d$ cm. The force expended at any instant during the upward motion of the piston is $\frac{1}{4}\,\pi\,d^2\,h$ grams, and as h is greatest at the end of the upward stroke this force is increasing. On the downward stroke the force is $\frac{1}{4}\,\pi\,d^2\,h'$ grams, since the column $P\,V$ balances the column $B\,V'$, leaving only $B\,A = h'$ to act; as $B\,A$ is greatest at the end of the down stroke this force is also increasing.

The piston is only one of many contrivances for producing rarefaction of air in a pump-tube; but since it is the most simple and most easily kept in repair, the piston-pump is generally preferred to any other.

260. The Fire-Engine.—This machine generally consists of one or more forcing pumps, with a regulating air-vessel, though the arrangement of parts is exceedingly varied. Fig. 176 will illustrate the principles of its construction. As the piston, P, ascends, the water is raised through the valve, V, by atmospheric pressure. As P descends, the water is driven through F into the air-vessel, M, whence by the condensed air it is forced out without interruption through the hose-pipe, L. The piston P' operates in the same way by alternate movements. The piston-rods are attached to a lever (not represented), to which the strength of several men can be applied at once by means of hand-bars called *brakes*.

FIG. 176.



The air-vessel may be attached to any kind of pump, whenever it is desired to render the stream constant.

261. Hero's Fountain.—The condensation in the air-vessel, from which water is discharged, may be produced by the weight of a column of water. An illustration is seen in Hero's fountain, Fig. 177. A vertical column of water from the vessel, *A*, presses into the air-vessel, *B*, and condenses the air more or less, according to the height of *A B*. From the top of this vessel an air-tube conveys a portion of the compressed air to a second air-vessel, *C*, which is nearly full of water, and has a jet-pipe rising from it. Since the tension of air in *C* is equal to that in *B*, a jet will be raised which, if unobstructed, would be equal in height to the compressing column, *A B*.

This plan has been employed to raise water from a mine in Hungary, and hence called "the Hungarian machine."

The principle of its application for this purpose may be understood from the annexed diagram.

FIG. 177.

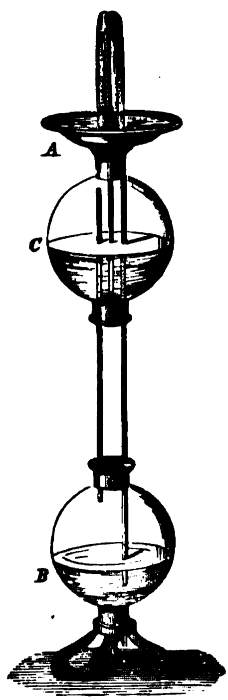
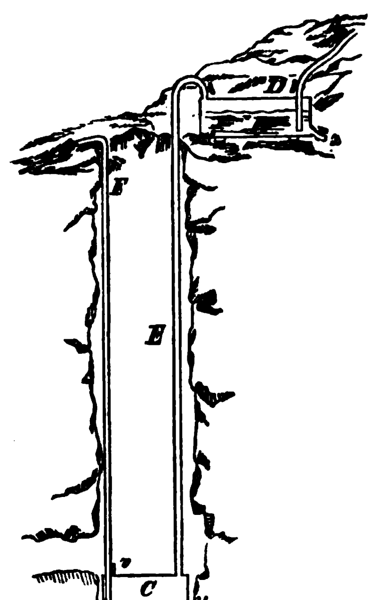


FIG. 178.

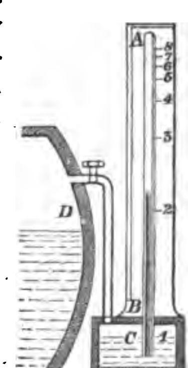


Let *A* represent a reservoir, or water supply, situated on high ground at an elevation above the mouth of the shaft greater than the depth of shaft to be drained. From this reservoir a pipe *D* (Fig. 178) passes to the *bottom* of a large and strong air chamber

B. From the *top* of the air chamber a pipe *E* passes to the *top* of a much smaller chamber *C*, at the bottom of the shaft, from the *bottom* of which passes the discharge pipe *F*, having a valve at *v*. Suppose the necessary valves to be supplied. Let all pipes and both chambers be filled with air only. Open the valve *y*, which will allow water from the mine to flow into *C*, driving out the air through *E* into *B* and out through the waste pipe *x*, which must also be open. Now close *y* and *x* and open *D*, which will permit water to flow from *A* into *B*, compressing the air in *B*, which pressure will be communicated through the air-pipe *E* to the surface of the water in *C*, driving it out through *F*. When *B* is full, or nearly full, of water, close *D*, open *x* and *y*, and thus allow water to flow into *C* and out of *B*. When *C* is full and *B* is empty, repeat the action as at first. For a shaft 100 feet deep, the air chamber *B* should be at least four times the capacity of *C*. If the height of the reservoir *A* above the mouth of the shaft be less than the depth of the shaft to be drained, the water must be raised by successive lifts.

262. Manometers.—These are instruments for measuring the tension of gases or vapors. The open manometer or "open mercurial gauge," as applied to the steam boiler, consists simply of a thick glass tube, standing vertical, both ends open, the lower end dipping into mercury contained in a closed cistern; a pipe connects the space above the mercury in the cistern with the steam space in the boiler. When the tension of the steam is equal to one atmosphere the pressure upon the mercury in the cistern will be balanced by the pressure of the air, transmitted through the open upper end of the tube. As the steam pressure increases, the mercury will rise in the tube, at a pressure of two atmospheres standing at about 30 inches, at three atmospheres at 60 inches, and so on. The pressure of steam is always given as so many pounds above one atmosphere; a boiler carrying 30 lbs. of steam, really has 45 lbs. internal pressure, 15 lbs. of which is counterbalanced by the pressure of the external air.

FIG. 179.



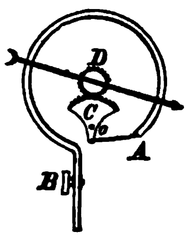
For high pressure a very long open mercurial gauge would be required; in such cases the closed manometer, or closed mercurial gauge may be used. This differs from the former in having the glass tube *A B* closed at the top, as represented in Fig. 179. In this instrument the graduation can be theoretically determined by Mariotte's law, considering that, if the tube be of

uniform diameter, a halving of the length of the enclosed air in the upper part of the tube would correspond to a doubling of the pressure exerted upon it. A correction must, of course, be applied for the weight of the column of mercury in the lower part of the tube.

As the uniformity of the bore of the tube cannot be assured, graduation by comparison with a standard gauge is the general method. Variations, caused by changes of temperature of the enclosed air, must be corrected by tables for the purpose. As the graduations crowd together near the top of the tube, as shown in the diagram, it has been proposed to substitute a tapering, conical tube for the cylindrical tube, giving it such proportions as to practically correct these inequalities.

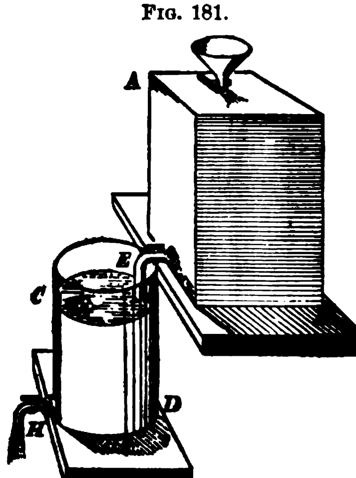
A metallic gauge, called from the inventor a Bourdon Gauge, or some modification of it, is in very common use. It consists of a flattened tube, bent as in Fig. 180, the closed end A being connected with a toothed sector C. When steam is admitted through the stop-cock at B, the curved tube *tends* to straighten as the pressure rises, and the motion of the end A in the direction of the arrow turns the sector C about its axis o, and by the teeth gives motion to the pinion D, which carries the index. These gauges are graduated by comparison with a standard mercurial gauge.

FIG. 180.



263. Apparatus for Preserving a Constant Level.—Let $A\ B$ (Fig. 181) be a reservoir which supplies a liquid to the vessel $C\ D$; and suppose it is desired to preserve the level at the point C in the vessel, while the liquid is discharged from it irregularly or at intervals. So long as the mouth of the pipe E is submerged in the liquid in $C\ D$, no air can enter the reservoir $A\ B$, and hence no liquid can flow from it; but when the liquid is drawn from $C\ D$ so that the level C falls, air will bubble up through the pipe E, displacing liquid in $A\ B$ till the end of the pipe E

FIG. 181.



is again closed; this action will be repeated as often as the level in $C\ D$ falls below C. The pipe E should be of greater cross-section than the pipe H, or else there must be a great head of water in $A\ B$, so that E may supply liquid faster than H can discharge it.

Problems.

1. The volume of the receiver of an air-pump is R, and that of the barrel is B: prove that if the density of the air in the receiver before exhaustion is D, the density after n complete strokes is,

$$D_n = \{R/(R + B)\}^n D.$$

2. The receiver of an air-pump has three times the volume of the barrel: find the density after ten complete strokes.

3. After three complete strokes the density of the air in the receiver of an air-pump was to its original density as 125 : 216: show that the volume of the receiver was five times that of the barrel.

4. A pump-handle is 1 m. long and is pivoted at 10 cm. from the end which is attached to the piston-rod. The spout is 5 m. above the level in the cistern, and the diameter of the piston is 10 cm. What force must be applied to the end of the handle in order to maintain a flow from the spout?

CHAPTER III.

THE ATMOSPHERE—ITS HEIGHT AND MOTIONS.

264. Virtual Height of the Atmosphere.—When two fluid columns are in equilibrium with each other, their heights are inversely as their specific gravities (Art. 194). The specific gravity of mercury is 10464 times that of the air at the ocean level. Therefore, if the air had the same density in all parts, its height would be found by the proportion,

$$1 : 10464 :: 2.5 : 26160 \text{ feet},$$

which is almost five miles. Hence, the quantity of the entire atmosphere of the earth is pretty correctly conceived of when we imagine it having the density of that which surrounds us, and reaching to the height of five miles.

265. Decrease of Density.—But the atmosphere is very far from being throughout of uniform density. The great cause of inequality is the decreasing weight of superincumbent air at increasing altitudes. The law of diminution of density, arising from this cause, is the following:

The densities of the air decrease in a geometrical as the altitudes increase in an arithmetical ratio. For, let us suppose the air to be divided into horizontal strata of equal thickness, and so thin that the density of each may be considered as uniform throughout. Let a be the weight of the whole column from the top to the earth, b the weight of the whole column above the lowest stratum, c that of the whole column above the second, &c. Then the weight of the lowest stratum is $a - b$, and the weight of the second is $b - c$, &c. Now the *densities* of these strata, and therefore their *weights* (since they are of equal thickness), are as the compressing forces; or,

$$a - b : b - c :: b : c;$$
$$\therefore a c - b c = b^2 - b c; \therefore a c = b^2;$$
$$\therefore a : b :: b : c;$$

in the same way, $b : c :: c : d$;

that is, the *weights* of the entire columns, from the successive strata to the top of the atmosphere, form a geometrical series; therefore, the *densities* of the successive strata, varying as the compressing forces, also form a geometrical series. If, therefore, at a certain distance from the earth, the air is twice as rare as at the surface of the earth, at twice that distance it will be four times as rare, at three times that distance eight times as rare, &c.

By barometric observations at different altitudes, it is found that at the height of three and a half miles above the earth the air is one-half as dense as it is at the surface. Hence, making an arithmetical series, with $3\frac{1}{2}$ for the common difference, to denote heights, and a geometrical series, with the ratio of $\frac{1}{2}$, to denote densities, we have the following:

Heights, $3\frac{1}{2}$, 7, $10\frac{1}{2}$, 14, $17\frac{1}{2}$, 21, $24\frac{1}{2}$, 28, $31\frac{1}{2}$, 35.
Densities, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$, $\frac{1}{512}$, $\frac{1}{1024}$.

According to this law, the air, at the height of 35 miles, is at least a thousand times less dense than at the surface of the earth. It has, therefore, a thousand times less weight resting upon it; in other words, only one-thousandth part of the air exists above that height.

266. Actual Height of the Atmosphere.—The foregoing law, founded on that of Mariotte, cannot, however, be applicable except to moderate distances. If it were strictly true, the atmosphere would be unlimited. But that is impossible on a revolving

body, since the centrifugal force must at some distance or other equal the force of gravity, and thus set a limit to the atmosphere; and that limit in the case of the earth is more than 20,000 miles high. The actual height of the atmosphere is doubtless far below this; for there can be none above the point where the repellency of the particles is less than their weight; and the repellency diminishes just as fast as the density, while the weight diminishes very slowly. The highest portions concerned in reflecting the sunlight are about 45 miles above the earth. But there is reason to believe that the air extends much above that height, probably 100 or 200 miles from the earth.

267. The Motions of the Air.—The air is never at rest. When in motion, it is called *wind*. The equilibrium of the atmosphere is disturbed by the unequal heat on different parts of the earth. The air over the hotter portions becomes lighter, and is therefore pressed upward by the cooler and heavier air of the less heated regions. And the motions thus caused are modified as to direction and velocity by the rotation of the earth on its axis.

268. The Trade Winds.—The most extensive and regular system of winds on the earth is known by the name of the *trade winds*, so called on account of their great advantage to commerce. They are confined to a belt about equal in width to the torrid zone, but whose limits are four or five degrees further north than the tropics.

In the northern half of this trade-wind zone the wind blows continually from the northeast, and in the southern half from the southeast. As these currents approach each other, they gradually become more nearly parallel to the equator, while between them there is a narrow belt of calms, irregular winds, and abundant rains.

The oblique directions of the trade winds are the combined effects of the heat of the torrid zone and the rotation of the earth. The cold air of the northern hemisphere tends to flow directly south, and crowd up the hot air over the equator. In like manner, the cold air of the southern hemisphere tends to flow directly northward. So that if the earth were at rest, there would be *north* winds on the north side of the equator, and *south* winds on the south side. But the earth revolves on its axis from west to east, and the air, as it moves from a higher latitude to a lower, has only so much eastward motion as the parallel from which it came. Therefore, since it really has a less motion from the west than those regions over which it arrives, it has relatively a motion *from the east*. This motion from the east, compounded with the motion from the north on the north side of the equator, and with that

from the south on the south side, constitutes the northeast and southeast tradewinds.

The limits of this system move a few degrees to the north during the northern summer, and to the south during the northern winter, but very much less than might be expected from the changes in the sun's declination.

In certain localities within the tropics the wind, owing to peculiar configurations of coast and elevations of the interior, changes its direction periodically, blowing six months from one point, and six months from a point nearly opposite. The *monsoons* of southern India are the most remarkable example.

269. The Return Currents.—The air which is pressed upward over the torrid zone must necessarily flow away northward and southward towards the higher latitudes, to restore the equilibrium. Hence, there are south winds in the upper air on the north side of the equator, and north winds on the south side. But these upper currents are also oblique to the meridians, because, having the easterly motion of the equator, they move faster than the parallels over which they successively arrive, so that a motion from the west is combined with the others, causing southwest winds in the northern hemisphere, and northwest in the southern. These motions of the upper air are discovered by observations made on high mountains, and in balloons, and by noticing the highest strata of clouds. It is to be borne in mind that although the atmosphere is more than 100 miles high, yet the lower half does not extend beyond three and a half miles above the earth (Art. 265).

270. Circulation Beyond the Trade Winds.—The upper part of the air which flows away from the equator cannot wholly retain its altitude, because of the diminishing space on the successive parallels. About latitude 30°, it is so much accumulated that it causes a sensible increase of pressure (Art. 243), and begins to descend to the earth. It is probable that some of the descending air still retains its oblique motion towards higher latitudes (for the prevailing winds of the northern temperate zone are from the southwest, and of the southern temperate zone from the northwest), while a part joins with the lower air which is moving towards the equator. Only so much of the rising equatorial mass can flow back to the polar regions as is needed to supply the comparatively small area within them. On account of the successive descent of the air returning from the equator, there is much less distinctness and regularity in the general circulation outside of the torrid zone than within it. Besides this, various local causes, such as mountain ranges, sea-coasts, and ocean cur-

rents, clear and cloudy skies, &c., mingle their effects with the more general circulation, and modify it in every possible way.

271. Land and Sea Breezes.—These are limited circulations over adjoining portions of land and water, the wind blowing from the water to the land in the day time, and in the contrary direction by night. When the sun begins to shine each day, it heats the land more rapidly than the water. Hence the air on the land becomes warmer and lighter than that on the water, and the surface current sets toward the land. By night the flow is reversed, because the land cools most rapidly, and the air above it becomes heavier than that over the water. These effects are more striking and more regular in tropical countries, but are common in nearly all latitudes.

272. A Current Through a Medium.—There are some phenomena relating to currents moving through a fluid, either of the same or a different kind, which belong alike to hydraulics and pneumatics; a brief account of these is presented here.

If a stream is driven through a medium, it carries along the adjoining particles by *friction* or *adhesion*. The experiment of Venturi illustrates this kind of action, as it takes place between the particles of water. A reservoir filled with water has in it an inclined plane of gentle ascent, whose summit just reaches the edge of the reservoir. A stream of water is driven up this plane with force sufficient to carry it over the top; but in doing so, it takes out continually some part of the water of the reservoir, and will in time empty it to the level of the lowest part of the stream. A stream of air through air produces the same effect, as may be shown by the flame of a lamp near the stream always bending toward it. In like manner, water through air carries air with it; when a stream of water is poured into a vessel of water, air is carried down in bubbles; and cataracts carry down much air, which as it rises forms a mass of foam on the surface. The strong wind from behind a high waterfall is owing to the condensation of air brought down by the back side of the sheet.

273. Ventilators.—If the stream passes across the end of an open tube, the air within the tube will be taken along with the stream and thus a partial vacuum formed, and a current established. It is thus that the wind across the top of a chimney increases the draught within. To render this effect more uniformly successful, by preventing the wind from striking the interior edge of the flue, appendages, called *ventilators*, are attached to the chimney top. A simple one, which is generally effectual, consists of a conical frustum surrounding the flue as in

Fig. 182, so that the wind, on striking the oblique surface, is thrown over the top in a curve, which is convex upward. The same mechanical contrivance is much used for the ventilation of public halls and the holds of ships. A horizontal cover may be supported by rods, at the height of a few inches, to prevent the rain from entering.

FIG. 182.

274. A Stream Meeting a Surface.—Though the moving fluid may be elastic, yet, when it meets a surface, it tends to *follow* it, rather than to rebound from it. This effect is partly due to adhesion, and partly to the resistance of the medium in which the stream moves. It will not only follow a plane or concave surface, but even one which is convex, provided the velocity of the current is not too great, or the curvature too rapid. A stream of air, blown from a pipe upon a plane surface, will extinguish the flame of a lamp held in the direction of the surface beyond its edge, while, if the lamp be held elsewhere near the stream, the flame will point toward the stream, according to Art. 272. Hence, snow is blown away from the windward side of a tight fence, and from around trees.

275. Diminution of Pressure on a Surface.—When a stream is thus moving along a surface, the fluid pressure on that surface is slightly diminished. This is proved by many experiments. If a curved vane be suspended on a pivot, and a stream of air be directed tangentially along the surface, it will move toward the stream, and may be made to revolve rapidly by repeating the blast at each half revolution. What is frequently called the *pneumatic paradox* is a phenomenon of the same kind. A stream of air is blown through the centre of a disk, against another light disk, which, instead of being blown off, is forcibly held near to it by the means. The pressure is diminished by all the radial streams along the surface contiguous to the other disk, and the full pressure on the outside preponderates. Another form of the experiment is to blow a stream of air through the bottom of a hemispherical cup, in which a light sphere is lying loosely. The sphere cannot be blown out, but, on the contrary, is held in, as may be seen by inverting the cup, while the blast continues. It appears to be for a reason of the same sort that a ball or a ring is sustained by a jet of water. It lies not on the *top*, but on the *side* of the jet, which diminishes the pressure on that side of the ball, so that the air on the outside keeps it in contact. The tangential force of the jet

causes the body to revolve with rapidity. A ball can be sustained a few inches high by a stream of air.

276. Vortices where the Surface Ends.—As a current reaches the termination of the surface along which it was flowing, a *vortex* or whirl is likely to occur in the surrounding medium behind the edge of the surface. Vortices are formed on water, whose flow is obstructed by rocks; and often when the obstructing body is at a distance below the surface, the whirl which is established there is communicated to the top, so that the vortex is seen, while its cause is out of sight. There is a depression at the centre, caused by the centrifugal force; and if the rotation is rapid, a spiral tube is formed, in which the air descends to great depths. These are called *whirlpools*. In a similar manner whirls are produced in the air, when it pours off from a surface. The eddying leaves on the leeward side of a building in a windy day often indicate such a movement, though it may have no permanency, the vortex being repeatedly broken up and reproduced.

277. Vortices by Currents Meeting.—But vortices are also formed by counteracting currents in an open medium. When an aperture is made in the middle of the bottom of a vessel, as the water runs toward it, the filaments encounter each other, and usually, though not invariably, they establish a rotary motion, and form a whirlpool. Vortices are a frequent phenomenon of the atmosphere, sometimes only a few feet in diameter, in other instances some rods or even miles in width. The smaller ones, occurring over land, are called *whirlwinds;* over water, *water-spouts*. They probably originate in currents which do not exactly oppose each other, but act as a *couple* of forces, tending to produce rotation (Art. 56).

The burning of a forest sometimes occasions whirlwinds, which are borne away by the wind, and maintain their rotation for miles. As the pressure in the centre is diminished by the centrifugal force, substances heavier than air, as leaves and spray, are likely to be driven up in the axis, and floating substances, as cloud, will for the same reason descend. The rising spray and the descending cloud frequently mark the progress of a vortex in the air, as it moves over a lake or the ocean. Such a phenomenon is called a water-spout.

PART IV.

ACOUSTICS.

CHAPTER I.

NATURE AND PROPAGATION OF SOUND.

278. Sound. — Vibrations. — The impression which the mind receives through the organ of hearing is called *sound*. But the same word is constantly used to signify that *progressive vibratory movement* in a medium by which the impression is produced, as when we speak of the velocity of sound.

The vibrations constituting sound are comparatively slow, and are often perceived by sight and by feeling as well as by hearing. For these reasons, the true nature of sound is investigated with far greater ease than that of light, electricity, &c. It is not difficult to discover that *vibrations* in the medium about us are essential to hearing; and these vibrations are always traceable to the body in which the sound originates. A body becomes a source of sound by producing an impulse or a series of impulses on the surrounding medium, and thus throwing the medium itself into motion. A single sudden impulse causes a *noise*, with very little continuance; an irregular and rapid succession of impulses a *crash*, or *roar*, or *continued noise* of some kind; but if the impulses are rapid and perfectly equidistant, the effect is a *musical sound*. In most cases of the last kind the impulses are vibrations of the body itself; and whatever affects these vibrations is found to affect the sound emanating from it; and if they are destroyed, the sound ceases.

If we rub a moistened finger along the edge of a tumbler nearly full of water, or draw a bow across the strings of a viol, we can procure sounds which remain undiminished in intensity as long as the operation by which they are excited is continued. In both cases the vibrations are visible; those of the tumbler are plainly seen as crispations on the water to which they are commu-

nicated; the string appears as a broad shadowy surface. If a wire or light piece of metal rests against a bell or glass receiver, when ringing, it will be made to rattle. If sand be strewed on a horizontal plate while a bow is drawn across its edge, the sand will be agitated, and dance over the surface, till it finds certain places where vibrations do not exist. Near an organ-pipe the tremor of the air is perceptible, and pipes of the largest size jar the seats and walls of an edifice. Every species of sound may be traced to impulses or vibrations in the sounding body.

279. Sonorous Bodies.—Two qualities in a body are necessary, in order that it may be sonorous. It must have a form favorable for vibrating movements, and sufficient elasticity.

The favorable *forms* are in general rods and plates, rather than very compact masses, like spheres and cubes; because the particles of the former are more free to receive lateral movements than those of the latter, which are constrained on every side. But even a thin lamina may have a form which allows too little freedom of motion, such as a spherical shell, in which the parts mutually support each other. If the shell be divided, the hemispheres are bell-shaped and very sonorous.

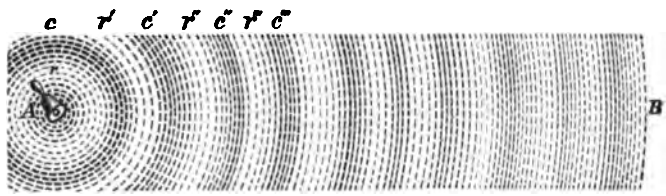
The *elasticity* of some materials is too imperfect for continued vibration; thus lead, in whatever form, has no sonorous quality.

280. Air as a Medium of Sound.—There must not only be a vibrating body, as a *source* of sound, but a medium for its *communication* to the organ of hearing. The ordinary medium is air. Let a bell mounted with a hammer and mainspring, so as to continue ringing for several minutes, be placed on a thick cushion under the receiver of an air-pump. The cushion, made of several thicknesses of woolen cloth, is necessary to prevent communication through the metallic parts of the instrument. As the process of exhaustion goes on, the sound of the bell grows fainter, and at length ceases entirely. From this experiment we learn that sound cannot be propagated through a vacant space, even though it be only an inch or two in extent.

281. Method of Propagation of Sound.—The vibrations of a medium in the transmission of sound are of the kind called *longitudinal, i.e.*, the particles move in the direction of the propagation. Let Fig. 183 represent the arrangement of some air particles at a certain instant. These particles are transmitting the sound vibrations of the bell at *A* to the point *B*. The bell an instant before has expanded. In expanding it has forced the air particles in contact with it to come into collision with their neighbors. Both

being perfectly elastic, they exchange velocities and rebound (Art. 96). In another instant the motion will be communicated to the

FIG. 183.



next series of particles. If the bell should become fixed at this instant, the impulse which it has just given would travel onward towards B with a definite velocity, to be discussed later. The impulse appears as a crowded condition of the particles. On the other hand, if the bell be not fixed, its return to its normal position will cause a rarefied condition of the air to follow after the condensed condition, with the same velocity. A particle, in aiding the transmission of these two conditions, travels first towards B, then back beyond its normal position towards A, and then returns to its position of equilibrium. The Figure represents an arrangement, resulting from a continued vibration of the bell, where c, c', c'', &c., are condensed conditions, and r, r', r'', &c., are rarefied conditions travelling towards B. This state of affairs constitutes a *wave* system, resulting from the small vibrations of the air particles.

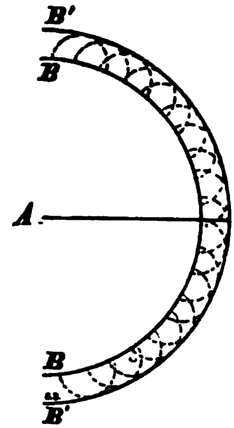
During the interval of time that it takes the bell to make a complete vibration, the impulse, which it first originated, travels through a definite distance, as c to c', or r to r'. Had the bell vibrated quicker, this distance would have been smaller. The distance between two successive condensed or rarefied conditions is termed the *wave length*. As in water waves (Art. 220) the wave length is the distance, measured on the line of propagation, between two successive particles in like phases.

The length of the path described by a single particle is termed the *amplitude of vibration*.

282. Huyghens's Principle.—A body sounding in open air sends out a system of waves, which consist of alternate, concentric, spherical condensations and rarefactions. Not only is this true, but, as Huyghens has shown, each individual particle sends out spherical waves by its vibrating; it is owing to this that the greater waves exist. For instance, a wave front from a sounding body, A

(Fig. 184) has, at a certain instant, reached $B\,B$. Every particle of this front is set in vibration and is a centre for further spherical waves. The number of these elementary spherical waves is very great, and they will evidently coalesce to form a new wave front $B'\,B'$.

FIG. 184.



This theory of the transmission of wave motions is called the *Huyghens's Principle.* Many phenomena of wave motion cannot be adequately explained without employing it.

283. Velocity of Sound in Air.—Sound occupies an appreciable time in passing through air. This is a fact of common observation. The flash of a distant gun is seen before the report is heard. Thunder usually follows lightning after an interval of many seconds; but if the electric discharge is quite near, the lightning and thunder are almost simultaneous. If a person is hammering at a distance, the perceptions of the blows received by the eye and the ear do not generally agree with each other; or if in any case they do agree, it will be observed that the first stroke seen is inaudible, and the last one heard is invisible; for it requires just the time between two strokes for the sound of each to reach us.

A long column of infantry, marching to the music of a single band, will have a vertical wave-like motion, since each rank steps to the music, and a given beat reaches the different ranks in succeeding periods of time.

Many careful experiments were made in the eighteenth century to determine the velocity of sound; but as the temperature was not recorded, they have but little value. During the present century, the velocity has been determined by several series of observations in different countries, and all reduced to the same temperature. The agreement between them is very close, and the mean of all is 332 meters, or 1090 feet per second at 0 C.

284. Newton's Formula.—Newton showed that the velocity of sound in air v, could be calculated from its elasticity E, and its density d by the formula

$$v = \sqrt{\frac{E}{d}}.$$

By the elasticity of the air is meant the following: Suppose the pressure on the air were changed by a small amount represented

by dp. The air would suffer a corresponding change of volume dv, according to Mariotte's law (Art. 237). The ratio of these two is the elasticity, or

$$E = \frac{dp}{dv}.$$

A little consideration will show that E increases as the pressure on the air increases. The density d also increases with the pressure, hence the *velocity of sound in air is not affected by any change in the barometric pressure.*

On the other hand, an increase of temperature will increase the volume of air, if it be under constant pressure, and hence decreases its density d. This increases the velocity v. *The velocity of sound in air increases about* 0.6 *meter for each degree Centigrade rise in temperature.*

When the air particles are condensed, in transmitting sound, there is an increase of temperature, owing to the latent heat made sensible. This increases the elasticity, and accordingly the velocity. In the following rarefied portion of the wave, cold is produced by the rarefaction. This cold decreases the elasticity of the rarefied portion, which has the same effect as increasing the elasticity of the condensed part. For it is the condensed portion that does the propagating, the rarefaction being the result of the condensation. *Hence both the heat of condensation and the cold of rarefaction increase the velocity of sound in air.*

Newton's formula does not take account of this and must be accordingly modified by the introduction of a constant. We have then

$$v = \sqrt{\frac{E}{d} 1.4.}$$

285. Velocity as Affected by the Condition of the Air and the Quality of the Sound.—*Wind*, of course, affects the velocity of sound by the addition or subtraction of its own velocity, estimated in the same direction, because it transfers the medium itself in which the sound is conveyed. This modification, however, is only slight, for sound moves ten times faster than wind in the most violent hurricane.

But other changes in the condition of the air produce little or no effect. Neither pressure, nor moisture, nor any change of weather, alters the *velocity* of sound, though they may affect its *intensity*, and therefore the distance at which it can be heard. Falling snow and rain *obstruct* sound, but do not *retard* it.

All *kinds* of sound—the firing of a gun—the blow of a hammer—the notes of a musical instrument, or of the voice, however

high or low, loud or soft, are conveyed at the same rate. That sounds of different pitch are conveyed with the same velocity was conclusively proved by Biot, in Paris, who caused several airs to be played on a flute at one end of a pipe more than 3000 feet long, and heard the same at the other end distinctly, and without the slightest displacement in the order of notes, or intervals of silence between them.

286. Other Gaseous Bodies as Media of Sound.—Let a spherical receiver, having a bell suspended in it, be exhausted of air till the bell ceases to be heard; then fill it with any gas or vapor instead of air, and the bell will be heard again. By means of an organ-pipe blown by different gases, it can be learned with what velocity sound would move in each kind of gas experimented upon, because the pitch of a given pipe depends upon the velocity of the waves, as will be seen hereafter.

From such experiments the following velocities at temperature 0° Centigrade have been deduced:

Carbonic acid, 856 ft. per second.

Oxygen, 1040 ft. per second.

Carbonic oxide, 1106 ft. per second.

Hydrogen, 4163 ft. per second.

From Newton's formula the velocity in gases varies directly as the square root of the elasticities and inversely as the square root of the densities; hence for the same elasticities, *i.e.*, the same pressures, the velocities should be inversely proportional to the square root of the densities. Oxygen being 16 times as heavy as hydrogen, the velocity of sound in the latter should be four times as great as in the former, which conclusion is confirmed by the facts given above. Momentary development of heat by compression produces, in all gaseous bodies, the effect of increasing the velocity of sound, as was shown in Art. 284 for air.

287. Liquids as Media.—Many experimenters have determined the circumstances of the propagation of sound in water. Franklin found that a person with his head under water could hear the sound of two stones struck together at a distance of more than half a mile. In 1826 Colladon made many careful experiments in the water of Lake Geneva. The results of these and other trials are principally the following:

1. Sounds produced in the air are very faintly heard by a person in water, though quite near. The kinetic energy of the light air particles is not sufficient to give a large motion to the heavier water particles. Sounds originating under water are feebly com-

municated to the air above, and in positions somewhat oblique are not heard at all, owing to reflection.

2. Sounds are conveyed by water with a velocity of 4700 feet per second, at the temperature of 8.3° C., which is more than four times as great as in air. The calculated and the observed velocity of sound in water agree so nearly with each other, that there appears to be no appreciable effect arising from heat developed by compression.

Calculated velocities are as follows:

Seine water, at 13° C. = 4714 ft. per second.
" " 30° C. = 5013 " "

Solution of calcic chloride, at 23° C. = 6493 ft. per second.

Sulphuric ether at 0° C. = 3801 ft. per second.

Hence sound travels with different velocities in different liquids; the velocity is greater in the liquid of greater density; the velocity is increased by increase of temperature.

3. Sounds conveyed in water to a distance, lose their sonorous quality. For example, the ringing of a bell gives a succession of short sharp strokes, like the striking together of two knife-blades. The musical quality of the sound is noticeable only within 600 or 700 feet. In air, it is well known that the contrary takes place; the blow of a bell-tongue is heard near by, but the continued musical note is all that affects the ear at a distance.

4. Acoustic *shadows* are formed; that is; sound passes the edges of solid bodies nearly in straight lines, and does not turn around them except in a very slight degree.

To enable the experimenter to hear distant sounds without placing himself under water, Colladon pressed down a cylindrical tin tube, closed at the bottom, thus allowing the acoustic pulses in the water to strike perpendicularly on the sides of the tube. In this way, the faintest sounds were brought out into the air. It appears to be true of sound as of light, that it cannot pass from a denser to a rarer medium at large angles of incidence, but suffers nearly a total reflection.

288. Solids as Media. — Solid bodies of high elasticity are the most perfect media of sound which are known. An iron rod—as, for instance, a lightning-rod—will convey a feeble sound from one extremity to the other, with much more distinctness than the air. If the ears are stopped, and one end of a long wire is held between the teeth, a slight scratch or blow on the remote end will sound very loud. The sound in this case travels through the wire and the bones of the head to the organ of hearing. The sound of earthquakes and volcanic eruptions is transmitted to

great distances through the solid earth. By laying the ear to the ground, the tramp of cavalry may be heard at a much greater distance than through the air.

289. Velocity in Solids. — Structure. — The velocity of sound in cast iron was estimated by Biot to be about 11000 feet per second—ten times greater than in air. He obtained this result by experiments on the aqueduct pipes in Paris. A blow upon one end was brought to an observer at the other end, 3000 feet distant, both by the iron and also by the air within it. The velocity in air being known, and the difference of time observed, the velocity in iron is readily calculated.

The following table is taken from Tyndall:

Name of Metal.	At 20° C.	At 100° C.	At 200° C.
Lead	4030	3951	——
Gold	5717	5640	5619
Silver	8553	8658	8127
Copper	11666	10802	9690
Iron	16822	17386	15483
Iron wire	16130	16728	——
Steel wire	16023	16443	——

As a rule the velocity in metals decreases with rise of temperature, but iron and silver are shown above to be exceptions to this general rule between the limits 20° C. and 100° C.

In one important particular solids differ from fluids, namely, in the fixed relations of the particles among themselves. These relations are usually different in different directions; hence, sound is likely to be transmitted more perfectly in some directions through a given solid than in others. The scratch of a pin at one end of a stick of timber seems loud to a person whose ear is at the other end. The sound is heard more perfectly in the direction of the grain than across it. In crystallized substances it is unquestionably true that the vibrations of sound move with different speed and with different intensity in the line of the axis, and in a line perpendicular to it.

The velocity in woods along the fibre is from about 11000 feet to 16000 feet; across the annual rings from 4500 feet to 6000; across the fibre, in the direction of the rings from about 2500 feet to 4500 feet, all of which velocities are approximate and depend upon the wood selected.

290. Mixed Media. — In all the foregoing statements it has been supposed that the medium was homogeneous; in other

words, that the material, its density, and its structure, continue the same, or nearly the same, the whole distance from the source of sound to the ear. If abrupt changes occur, even a few times, the sound is exceedingly obstructed in progress. When the receiver is set over the bell on the pump plate, the sound in the room is very much weakened, though the glass may not be one-eighth of an inch in thickness, and is an excellent conductor of sound. The vibrations of the internal air are very imperfectly communicated to the glass, and those received by the glass pass into the air again with a diminished intensity. If a glass rod extended the whole distance from the bell to the ear, the sound would arrive in less time, and with more loudness, than if air occupied the whole extent. For a like reason, walls, buildings, or other intervening bodies, though good conductors of sound themselves, obstruct the progress of sound in the air. When the texture of a substance is loose, having many alternations of material, it thereby becomes unfit for transmitting sound. It is for this reason that the bell-stand, in the experiment just referred to, is set on a cushion made of several thicknesses of loose flannel, that it may prevent the vibrations from reaching the metallic parts of the pump. The waves of sound, in attempting to make their way through such a substance, continually meet with new surfaces, and are *reflected* in all possible directions, by which means they are broken up into a multitude of crossing and interfering waves, and are mutually destroyed. A tumbler, nearly filled with water, will ring clearly; but if filled with an effervescing liquid, it will lose all its sonorous quality, for the same cause as before. The alternate surfaces of the liquid and gas, in the foam, confuse the waves, and deaden the sound.

291. Intensity of Sound.—*The intensity or loudness of sound depends upon the amplitude of the vibrations of the particles conveying it.* To obtain a loud tone from a piano its keys must be struck with great force, thus increasing the amplitude of vibration of the strings.

As has been shown, when a sound is produced in open air the wave-motion is propagated in all directions alike, the entire system of waves around the point where sound originates consisting of spherical strata of air alternately condensed and rarefied. As the quantity set in motion in these successive layers increases with the square of the distance, the amount of motion communicated to each particle must diminish in the same ratio. Hence, the intensity of sound varies inversely as the square of the distance.

Intensity depends upon the density of the air in which the sound is produced, but not upon that of the air through which it

is transmitted. A sound which could be heard in water at a distance of 23 feet would be audible in air at only 10 feet. The report of a cannon, fired upon a mountain side, heard by a person in the rare air of the summit, would have the same intensity as the same report heard in the valley below; but a gun fired in the rare air of the summit might not be heard in the valley, while a report in the valley would be heard distinctly upon the summit, the intensity depending upon the density of the medium in which the sound is produced, as stated above.

Intensity is modified by motion of the air. In still air sound is more perfectly transmitted than when air currents exist. In case of winds sound is more intense, for a given distance, in the direction of the wind than in the contrary direction.

Sound is strengthened by *sympathetic* vibrations of other bodies than that which first produced the pulses. A vibrating string produces a sound scarcely audible; but when it vibrates upon a sounding box, the sympathetic vibrations of the latter are communicated to the air and a loud sound results. A vibrating tuning-fork held in the hand cannot be heard; the same fork caused to vibrate over the mouth of a cylinder closed at one end, and of a length equal to one-fourth of the wave length corresponding to the pitch of the fork will give a very loud sound. In all such cases the kinetic energy of the vibrating source is used up more rapidly when producing the loud sound, for it has to set a greater mass in vibration.

292. Diffusion of Sound.—Sound produced in the open air tends to spread equally in all directions, and will do so whenever the original impulses are alike on every side. But this is rarely the case. In firing a gun, the first impulse is given in one direction, and the sound will have more intensity, and be heard further in that direction than in others. It is ascertained by experiment, that a person speaking in the open air can be equally well heard at the distance of 100 feet directly before him, 75 feet on the right and left, and 30 feet behind him; and therefore an audience, in order to hear to the best advantage, should be arranged within limits having these proportions. But, as will be seen hereafter, this rule is not applicable to the interior of a building.

Sound is also heard in certain directions with more intensity, and therefore to a greater distance, if an obstacle prevents its diffusion in other directions. On one side of an extended wall sound is heard further than if it spread on both sides; still further, in an angle between two walls; and to the greatest distance of all, when confined on four sides, and limited to one direction, as in a long tube. The reason in these several cases is obvious; for a given

force can produce a given amount of motion; and if the motion is prevented from spreading to particles in some directions, it will reach more distant ones in those directions in which it does spread. Speaking-tubes confine the movement to a slender column of air, and therefore convey sound to great distances, and are on this account very useful in transmitting messages and orders between remote parts of manufacturing edifices and public houses.

CHAPTER II.

REFLECTION, REFRACTION, AND INFLECTION OF SOUND.

293. Reflection of Sound.—When sound waves arrive at a boundary between two media three things may occur:

(1.) The particles of the second medium may move with equal facility, as those of the first. The wave will in this case proceed as if the first medium were continuous.

(2.) The particles of the second medium may move with less facility. In this case the particles of the first medium will rebound. The path described after rebounding will be the same as before, if the boundary is perpendicular to it. If not perpendicular, it will follow the laws laid down in Art. 98. The continuity of the phases is not destroyed—simply the direction of the line of propagation is changed, and the wave continues after the change, and is said to have been *reflected.*

(3.) The particles of the second medium may move with greater facility than those of the first. In this case, the particles of a condensed portion in the first medium may be said to have come into collision with less resistance than they had expected, and accordingly to have moved farther than they intended. Their followers do the same, and the result is a reflected wave having a discontinuity of phases. The change in the direction of propagation is the same as in case 2, but there is a *loss of half a wave length* at the boundary.

These facts are as true for light waves or any waves as for those of sound.

Sound waves in air are reflected from a solid barrier, inasmuch as the air is perfectly elastic, according to the law that

The lines of propagation of the incident and reflected waves make equal angles on opposite sides of the perpendicular to the reflecting surface.

294. Echoes.—When sound is so distinctly reflected from a surface that it seems to come from another source, it is called an

echo. Broad and even surfaces, such as the walls of buildings and ledges of rock, often produce this effect. According to the law just given, a person can hear the echo of his own voice only by standing in a line which is perpendicular to the echoing surface. In order that one person may hear the echo of another's voice, they must place themselves in lines making equal angles with the perpendicular.

The interval of time between a sound and its echo enables one to judge of the distance of the surface, since the sound must pass over it twice. Thus, if at the temperature of 23° C., the echo of the speaker's voice reaches him in two seconds after its utterance, the distance of the reflecting body is about 1130 feet, and in that proportion for other intervals. And he can hear a distant echo of as many syllables as he can pronounce while sound travels twice the distance between himself and the echoing surface.

The ear can recognize about nine successive sounds in one second; two sounds separated by less than one-ninth of a second blend and produce confusion; therefore the distance from the speaker to the reflecting surface at temperature 0° C. must not be less than $\frac{1090}{9} \div 2 = 60.5$ ft. in order that an echo of a sharp sound may be heard. For articulate sound at ordinary temperatures the distance may be about 112,5 ft.

295. Simple and Complex Echoes.—When a sound is returned by one surface, the echo is called *simple;* it is called *complex* when the reflection is from two or more surfaces at different distances, each surface giving one echo. Thus, a cannon fired in a mountainous region is heard for a long time echoed on all sides, and from various distances.

A complex echo may also be produced by two parallel walls, if the hearer and the source of sound are both situated between them. The firing of a pistol between parallel walls a few hundred feet apart has been known to return from 30 to 40 echoes before they became too faint to be heard. The rolling of thunder is in part the effect of reverberation between the earth and the clouds. This is made certain by the observed fact that the report of a cannon, which in a level country and under a clear sky is sharp and single, becomes in a cloudy day a prolonged roar, mingled with distant and repeated echoes. But the peculiar inequalities in the reverberations of thunder are doubtless due in part to the irregularly crinkled path of the electric spark. A discharge of lightning occupies so short a time, that the sound may be considered as starting from all points of its track at once. But that track is full of large and small curves, some convex and some concave to

the ear, and at a great variety of distance; and all points which are at equal distances would be heard at once. Hence the original sound comes to the hearer with great irregularity, loud at one instant and faint at another. These inequalities are prolonged and intensified by the echoes which take place between the clouds and the earth.

296. Concentrated Echoes.—The divergence of sound from a plane surface continues the same as before, that is, in spherical waves, whose centre is at the same distance behind the plane as the real source is in front. But concave surfaces in general produce a concentrating effect. A sound originating in the centre of a hollow sphere will be reflected back to the centre from every point of the surface. If it emanates from one focus of an ellipsoid, it will, after reflection, all be collected at the other focus. So, if two concave paraboloids stand facing each other, with their axes coincident, and a whisper is made at the focus of one, it will be plainly heard at the focus of the other, though inaudible at all points between. In the last case the sound is twice reflected, and passes from one reflector to the other in parallel lines. All these effects are readily proved from the principle that the angles of incidence and reflection are equal.

The speaking-trumpet and the ear-trumpet have been supposed by many writers to owe their concentrating power to multiplied reflections from the inner surface. But a part of the effect, and sometimes the whole, is doubtless due to employing the energy in one direction, by preventing lateral diffusion, till the intensity is greatly increased.

Concave surfaces cause all the curious effects of what are called *whispering galleries*, such as the dome of St. Paul's, in London. In many of these instances, however, there seems to be a continued series of reflections from point to point along the smooth concave wall, which all meet simultaneously (if the curves are of equal length) at the opposite point of the dome; for the whisperer places his mouth, and the hearer his ear, close to the wall, and not in the focus of the curve. The *Ear of Dionysius* was probably a curved wall of this kind in the dungeons of Syracuse. It is said that the words, and even the whispers, of the prisoners were gathered and conveyed along a hidden tube to the apartment of the tyrant. The sail of a ship when spread, and made concave by the breeze, has been known to concentrate and render audible to the sailors the sound of a bell 100 miles distant.

297. Resonance of Rooms.—If a rectangular room has smooth, hard walls, and is unfurnished, its reverberations will be

loud and long-continued. Stamp on the floor, or make any other sudden noise, and its echoes passing back and forth will form a prolonged musical note, whose pitch will be lower as the apartment is larger. This is called the *resonance* of the room. Now, let furniture be placed around the walls, and the reverberations will be weakened and less prolonged. Especially will this be the case if the articles be of the softer kinds, and have irregular surfaces. Carpets, curtains, stuffed seats, tapestry, and articles of dress have great influence in destroying the resonance of a room. The appearance of an apartment is not more changed than is its resonance by furnishing it with carpet and curtains. The blind, on entering a strange room, can, by the sound of the first step, judge with tolerable accuracy of its size and the general character of its furniture.

The reason why substances of loose texture do not reflect sound well, is that they are not homogeneous—the waves are reflected in all directions by successive surfaces, interfere with each other, and are destroyed.

293. Halls for Public Speaking.—In large rooms, such as churches and lecturing halls, all echoes which can accompany the voice of the speaker syllable by syllable, are useful for increasing the volume of sound; but all which reach the hearers sensibly later, only produce confusion. It is found by experiment that if a sound and its echo reach the ear within *one-sixteenth* of a second of each other, they seem to be one. Hence, this fraction of time is called the *limit of perceptibility*. Within that time an echo can travel about 70 feet more than the original sound, and yet appear to coincide with it. If an echoing wall, therefore, is within 35 feet of the speaker, each syllable and its echo will reach every hearer within the limit of perceptibility. The distance may, however, be increased to 40 or even 50 feet without injury, especially if the utterance is not rapid. Walls intended to aid by their echoes should be smooth, but not too solid; plaster on lath is better than plaster on brick or stone; the first echo is louder, and the reverberations less. Drapery behind the speaker deprives him of the aid of just so much echoing surface. A lecturing hall is improved by causing the wall behind the speaker to change its direction, on the right and left of the platform, at a very obtuse angle, so as to exclude the rectangular corners from the room. The voice is in this way more reinforced by reflection, and there is less resonance arising from the parallelism of opposite walls. Paneling, and any other recesses for ornamental purposes, may exist in the reflecting walls without injury, provided they are not curved. The ceiling should not be so high that the reflection from it would be delayed beyond the limit of perceptibility. Concave

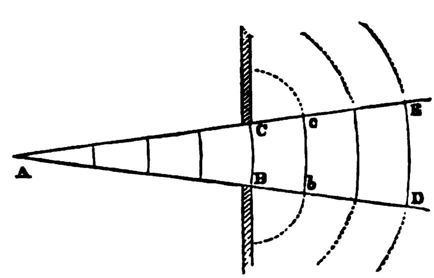
surfaces, such as domes, vaults, and broad niches, should be carefully avoided, as their effect generally is to concentrate all the sounds they reflect. An equal diffusion of sound throughout the apartment, not concentration of it to particular points, is the object to be sought in the arrangement of its parts.

As to distant parts of a hall for public speaking, the more completely all echoes from them can be destroyed, the more favorable is it for distinct hearing. It is indeed true that if a hearer is within 35 feet of a wall, however remote from the speaker, he will hear a syllable, and its echo from that wall, as one sound; but to all the audience at greater distances from the same wall, the echoes will be perceptibly retarded, and fall upon subsequent syllables, thus destroying distinctness. The distant walls should, by some means, be broken up into small portions, presenting surfaces in different directions. A gallery may aid in effecting this; and the seats of the gallery and of the lower floor may rise rapidly one behind another, so that the audience will receive directly much of the sound which would otherwise go to the remote wall, and be reflected. Especially should no large and distant surfaces be *parallel* to nearer ones, since it is between parallel walls that prolonged reverberation occurs.

299. Refraction of Sound.—It has been ascertained by experiment that sound, like light, may be *refracted*, or bent out of its rectilinear course by entering a substance of different density. If a large convex lens be formed of carbonic acid gas, by inclosing it in a sphere of thin india-rubber, a feeble sound, like the ticking of a watch, produced on one side, will be concentrated to a focal point on the other. In this case, the several diverging rays of sound are refracted toward each other on entering the sphere, and still more on leaving it, so that they are converged to a focus.

300. Inflection of Sound.—If air-waves are allowed to pass through an opening in an obstructing wall, they are not entirely confined within the radii of the wave-system produced through the opening, but spread with diminished intensity in lateral directions. The particles near the edges of the opening, as B and C (Fig. 185) are (Art. 282) sources of sound; and if they be made centres of concentric

FIG. 185.



spheres, whose radii are equal to the length of the wave, $B\,b$, or $C\,c$, and its multiples, then these spherical surfaces will represent the lateral systems of waves which are diffused on every side of the direct beam, $B\,D$, $C\,E$. But the sound is in general more feeble as the distance from $B\,D$, or $C\,E$, is greater, and in certain points is destroyed by interference. This spreading of sound in lateral directions is called the *inflection* of sound.

CHAPTER III.

MUSICAL SOUNDS AND MODES OF PRODUCING THEM.

301. Characteristics of Musical Sounds.—When the impulses of a sounding body upon the air are equiperiodic, and of sufficient frequency, they produce what is termed a musical sound. In most cases these impulses are the isochronous vibrations of the body itself, but not necessarily so; it is found by experiment that blows or pulses, of any species whatever, if they are more than about 15 or 20 per second, and possess the property of *isochronism*, cause a musical tone. For example, the snapping of a stick on the teeth of a metallic wheel would seem as unlikely as anything to produce a musical sound; but when the wheel is in rapid motion, the succession causes a pure musical note.

Musical tones possess three fundamental independent characteristics depending upon the character of the vibrations of the sounding body:

(1.) *Pitch*, depending on the *frequency* of vibrations.

(2.) *Intensity* or *Loudness*, depending on the *amplitude* of the vibrations.

(3.) *Quality* or *Timbre*, depending on the *form* or *shape* of the vibrations.

302. The Pitch of Musical Sounds.—What is called the *pitch* of a musical sound, or its degree of acuteness, is owing entirely to its rate of vibration. In comparing one musical sound with another, if the number of vibrations per second is greater, the sound is more acute, and is said to be of a *higher pitch;* if the vibrations are fewer per second, the sound is graver, or of a *lower pitch.*

303. The Monochord or Sonometer.—If a string of uniform size and texture is stretched on a box of thin wood, by means of a pulley and weight, the instrument is called a *monochord*, and

is useful for studying the laws of vibrations in musical sounds. The sound emitted by the vibrations of the whole length of the string is called its *fundamental* sound.

If the string be drawn aside from its straight position, and then released, one component of the force of tension urges every particle back towards its place of rest; but the string passes beyond that place, on account of the momentum acquired, and deviates as far on the other side; from which position it returns, for the same reason as before, and continues thus to vibrate till obstructions destroy its motion. By the use of a bow, the vibrations may be continued as long as the experimenter chooses.

The pitch of the fundamental sound of musical *strings* is found by experience to depend on three circumstances; the *length* of the string—its *weight* or quantity of matter—and its *tension*. The tone becomes more acute as we increase the tension, or diminish either the length or the weight. The operation of these several circumstances may be seen in a common violin. The pitch of any one of the strings is raised or lowered by turning the screw so as to increase or lessen its tension; or, the tension remaining the same, higher or lower notes are produced by the same string, by applying the fingers in such a manner as to shorten or lengthen the string which is vibrating; or, both the tension and the length of the string remaining the same, the pitch is altered by making the string larger or smaller, and thus increasing or diminishing its weight.

304. Time of a Complete Vibration.—The mathematical formula for the time of a complete vibration is

$$T = 2\,l\left(\frac{w}{g\,t}\right)^{\frac{1}{2}},$$

in which T is the time, in seconds, of a vibration; l = length of the string; w = the weight of a unit-length of the string; t = the tension, and g = the acceleration of gravity. In applying the formula l and g must be in the same unit, either both in centimeters or both in feet, and also w and t in the same unit, both in grams or both in pounds.

The constant factors, g and 2 being omitted, the variation may be expressed thus:

$$T \propto \frac{l\sqrt{w}}{\sqrt{t}}; \text{ that is,}$$

The time of a vibration varies as the length of the string multiplied by the square root of its weight per unit length, and divided by the square root of its tension.

As the *distance* of the string from its quiescent position does

not form an element of the algebraic expression for the time of a vibration, it follows that the time is independent of the amplitude. Hence, as in the pendulum, the vibrations of a string, fixed at both ends, are performed in equal times, whether the amplitude of the vibrations be greater or smaller. It is on this account that the pitch of a string does not alter, when left to vibrate till it stops. The excursions from side to side grow less, and therefore the sound more feeble, till it ceases; but the *rate* of vibration, and therefore the pitch, remains the same to the last. This property of *isochronism*, independent of extent of excursion, is common to sounding bodies generally.

305. The Number of Vibrations in a Given Time.—The greater is the time of one vibration, the less will be the number of vibrations in a given time; that is, if N represents the number, $N \propto \frac{1}{T}$, $\therefore N \propto \frac{\sqrt{t}}{l\sqrt{w}}$. If t and w are constant, $N \propto \frac{1}{l}$; if l and t are constant, $N \propto \frac{1}{\sqrt{w}}$; and if l and w are constant, $N \propto \sqrt{t}$; that is,

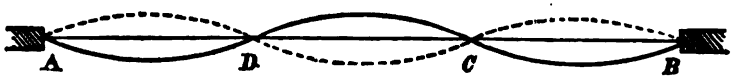
1. *The number of vibrations varies inversely as the length.*
2. *The number of vibrations varies inversely as the square root of the weight of the string.*
3. *The number of vibrations varies as the square root of the tension.*

Thus, the number of vibrations in a second may be doubled, either by *halving* the length of the string or by making its weight *one-fourth as great*, or, finally, by making its tension *four times as great.*

306. Vibrations of a String in Parts.—The string of a monochord may be made to vibrate in parts, the points of division remaining at rest; and this mode of vibration may even coexist with the one already described. Of course the sound produced by the parts will be on a higher pitch, since they are shorter, while the tension and the weight per unit length remain unaltered. It is a noticeable fact that the parts are always such as will exactly measure the whole without a remainder. Hence the vibrating parts are either halves, thirds, fourths, or other aliquot portions. The sounds produced by any of these modes of vibration are called *harmonics*, for a reason which will appear hereafter. Suppose a string (Fig. 186) to be stretched between A and B, and that it is thrown into vibration in three parts. Then while $A\ D$ makes its excursion on one side, $D\ C$ will move in the opposite direction, and $C\ B$ the same as $A\ D$; and when one is reversed, the others

are also, as shown by the dotted line. In this way D and C are kept at rest, being urged toward one side by one portion of string, and toward the opposite by the next portion. But the string may at the same time vibrate as a whole; in which case D and C will have motion to each side of their former places of rest, while relatively to them the three portions will continue their movements

FIG. 186.



as before. The points C and D are called *nodes*; the parts $A\ D$, $D\ C$, and $C\ B$, are called *ventral segments*. By a little change in the quickness of the stroke, the bow may be made to bring from the monochord a great number of harmonic notes, each being due to the vibrations of certain aliquot parts of the string. By confining a particular point, however, at the distance of $\frac{1}{2}$, $\frac{1}{3}$, or other simple fraction of the whole from the end, the particular harmonic belonging to that mode of division may be sounded clear, and unmingled with the others.

307. Longitudinal Vibrations of Strings.—A string may vibrate in the direction of its length, in consequence of its tractional elasticity (Art. 100). Such vibrations can be produced by rubbing the stretched string quickly with resined leather in the direction of its length.

The fundamental note thus produced is of much higher pitch than that of the same string caused to vibrate transversely.

By experimenting as in the case of transverse vibrations it may be shown that the number of vibrations in a given time is inversely proportional to the length of the string.

By altering the tension within the limits of elasticity of the substance no change of pitch is produced, longitudinal vibrations differing in this respect from transverse.

Changing the thickness or weight, the material being the same, does not alter the pitch, another difference to be noted between transverse and longitudinal vibrations.

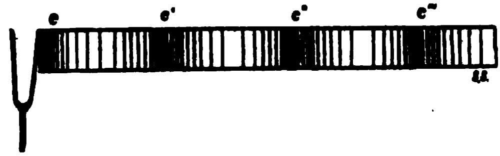
Two wires of different material but of the same length will give different notes.

Now if two wires of the same length but of different materials be used, that which transmits the sound pulse with the greatest velocity will give the highest note, since the number of vibrations per second due to this greater velocity of transmission will be greater. If two wires of different materials be so adjusted as to

length as to give the same note, the ratio of their lengths is the ratio of the velocities of transmission of sound in the two substances.

308. Pitch or Frequency determined by Wave Length.—We have seen that air, while transmitting sound, is arranged in alternate layers of condensation and rarefaction. These are shown in Fig. 187, the sounding body being a tuning-fork. The distance between two successive condensations, as *c c'*, is called a wave length. This wave length depends upon two things—the *velocity* of sound propagation in air and the *frequency* of the vibrations at the source. Now, as the velocity of propagation is nearly constant, the *wave length may be taken as a measure of the frequency or pitch.* The shorter the wave length the higher the pitch. The same pitch will always have the same wave length, whatever be the sounding source. If, for instance, the prongs of a tuning-fork, giving 256 vibrations per second, move away from each other, they will produce a condensation of air in their vicinity. This condensation will travel away from the fork to a distance of 1090 feet in 1 second, providing the temperature be 0° C. In the meantime, however, the fork has vibrated and has sent out 255 other condensations. These follow each other, being separated by equal distances. Evidently 1090 divided by this distance equals 256, or the frequency of the vibrations of the sounding body. The wave length is then about 4.26 feet, or about 52 inches.

FIG. 187.



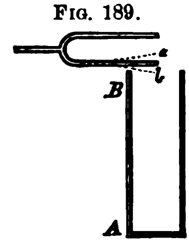
309. Resonant Cavities.—If the fork, just employed, be excited and held over the top of a 15-inch jar (Fig. 188), and if water be slowly poured into the jar, it will be found that when the water has reached a certain height the loudness of the sound given off by the fork will be greatly increased. This reinforcement is caused by the vibration of the air column above the water. It seems that air is not only adapted to transmit sound, but also to serve as a source of sound. If, now, more water be poured in, the reinforcement will cease. It appears that a column of air of definite length is capable of responding to, or being set in vibration by, a sounding body having a 52-inch wave length. Measurement will show that this is 13 inches. Had a fork of different pitch been used, the length of the resonant air column would have been different. In

each case, however, the length of the air column would be *one-quarter of the wave length* to which it responds.

FIG. 188.

In order to understand the relation existing between wave lengths and the lengths of responding resonators let us refer to Fig. 189. The lower prong of the fork requires $\frac{1}{512}$ of a second to move from the position *a* to *b*. During this interval the pulse of condensation, caused by the initial movement at *a*, would travel through 26 inches, if the fork were in open air. In the resonating jar the pulse will travel through the same distance, but will be reflected at the surface of the water. It will move through 13 inches, be reflected at the water, and retrace the 13 inches, arriving at *b* in time to coincide in direction with the movement of the prong back to *a*. In the same manner, the rarefied impulse, caused by the upward movement of the prong, will traverse the air column and return to *a* at the proper instant. The whole air column vibrates in the same time as the fork, and, acting as an additional source of sound, increases the loudness.

FIG. 189.



310. Stationary Sound Waves.—The vibrations of the air particles in a resonator are said to form *stationary waves*, *i.e.*, some of the particles which convey the wave do not move. The position of these particles is termed a *node*. In the case of the resonator just discussed, the reflecting water surface is a node, for the air particles in its proximity do not move when transmitting the wave. For instance, suppose a condensation is suffering reflection. The incoming portion of the wave tends to crowd the particles into the water, but the reflected portion tends to drive the particles away from the water, and, under their combined efforts, the particles remain at rest. They are equally stationary, whatever be the phase of the wave undergoing reflection. The particles at the open portion of the resonator, however, suffer a maximum displacement and correspond to the ventral segments of vibrating strings (Art. 306).

311. Stopped Organ Pipes.—If a stream of air be blown over the top of our resonator, a tone of 256 vibrations per second will result. A fluttering of the air particles at the open end is caused by the stream, and the whole air column picks out those movements which are synchronous with its own rate, and allows them to set it into vibration. Stopped mouth organ pipes are constructed on this principle. Fig. 190 represents a section of such a pipe. Wind from a wind-chest entering the channel *i* and issuing at the mouth *o* strikes against the sharp lip *b*. The sharp edge causes a chattering much the same as an oar causes ripples in a river current. The air column in the pipe is thus set in motion and gives a tone whose wave length is four times the length of the column. That it is the air, and not the pipe itself, which is the source of sound, is proved by using pipes of various materials—the most elastic and the most inelastic—as glass, wood, paper, and lead; if they are of the same form and size, the tone has, in each case, the same *pitch*.


FIG. 190.

312. Open Organ Pipes.—If the stopped end of a stopped pipe be removed (Fig. 191) it will still give a musical tone. The wave length, however, will be but half what it was before. The pitch shows this. The sound is caused by the stationary waves of the air column, and, as both ends of the column are in contact with the open air, these parts are ventral segments. Between these points there must be at least one node. In case the pipe is giving its fundamental tone there is one node only, and this is halfway up the pipe.

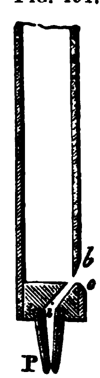


FIG. 191.

313. Vibrations of a Column of Air in Parts—Nodes.—If air enters an organ pipe under a much greater pressure than that intended for the pipe, this will give a tone of higher pitch than its fundamental tone. If the pressure be now gradually reduced, a point will be reached when the pipe will yield the two tones simultaneously. The explanation of this is that a column of air can vibrate in parts the same as a string (Art. 306). Also it can vibrate as a whole and in parts at the same time.

In the case of a stopped pipe the vibrations are limited only in that the stopped end must always be a node and the other end must be a ventral segment; a node because no motion of the air particles is possible at the stop, and a ventral segment because

all vibrations are generated at the mouth. When a stopped pipe gives its fundamental tone it has but one node and one ventral segment. The next higher tone which can be produced has three times the frequency of vibration, for the necessity that the top remain a node and the bottom a ventral segment requires the insertion of one extra node and one extra ventral segment. This division is shown in Fig. 192, where nodes and segments are represented by N and V respectively. The wave length has been reduced to one-third its former value. The next forced tone requires the insertion of two nodes and two segments. Their arrangement is shown in Fig. 193. The wave length is one-fifth that of the fundamental. Thus it can be seen that *a stopped pipe can be forced to give tones whose frequencies of vibrations are as the odd numbers,* 1, 3, 5, &c.

FIG. 192.

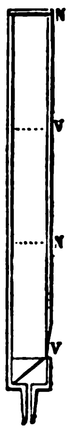
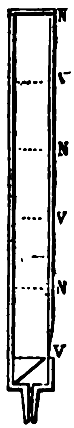


FIG. 193.



In the case of an open pipe, both ends must remain ventral segments. The first forced tone requires that the arrangement shall be as shown in Fig. 194. The wave length is half its fundamental value, and the frequency is accordingly doubled. The next forcing gives one-third the original wave length (Fig. 195) and the frequency is trebled. *An open pipe can be forced to yield tones whose frequencies are as* 1, 2, 3, 4, &c. The position of nodes in pipes may be readily shown by introducing a ring, R (Fig. 194), over which is stretched a thin membrane. If this ring, suspended from a cord, be lowered into the open upper end of a sounding pipe of glass, or one which has one transparent face, it will make a rattling or fluttering noise, or will show that it is vibrating by the motion of grains of sand sprinkled upon it; this vibration decreases in intensity as the disc is gradually lowered, until, when the disc reaches the place of a node, the vibration ceases and the sand remains at rest. Thus the place of each of any number of nodes may be determined experimentally. Fine silica powder in a sounding pipe held horizontally will also mark the segments and nodes very beautifully.

FIG. 194.

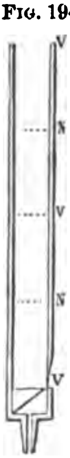


FIG. 195.

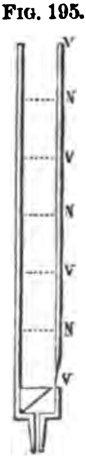
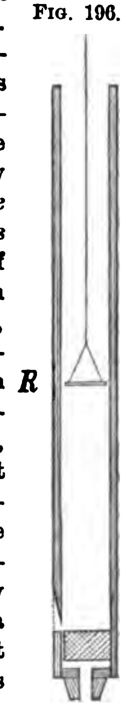


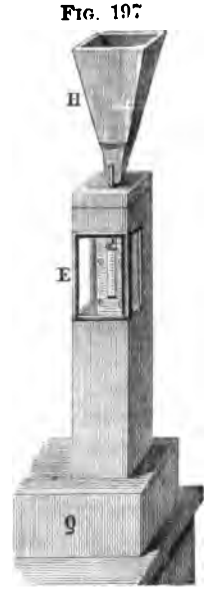
FIG. 196.



314. Modes of Exciting Vibrations in Pipes.—There are two methods of making the air column in a pipe to vibrate: one by a stream of air blown across an orifice in the pipe, the other by an elastic plate called a *reed*. A familiar example of the first is the *flute*. A stream of air from the lips is directed across the *embouchure*, so as just to strike the opposite edge; this excites a wave in the tube. A large proportion of the pipes of an organ are made to produce musical tones essentially in the same way as the flute, and are called *mouth-pipes*.

FIG. 198.

FIG. 197



Reeds are of two kinds, the *free* reed and the *beating* reed. The former passes to and fro through an aperture without touching its sides. Harmoniums and accordeons employ free reeds. The beating reed falls against the sides of an aperture, which it periodically opens and closes. The clarinet is excited by such a one. In that instrument the reed is often made of wood; when the air is blown past its edge into the tube, the reed is thrown into vibration, and by it the column of air. What are called the reed pipes of the organ are constructed on the same principle, but the reeds are metallic. An example is seen in Fig. 197, which represents a model of the reed pipe, made to show the vibrations through the glass walls at *E*. A chimney, *H*, is usually attached, sometimes of a form (as in the figure) to increase the loudness of the sound, and sometimes of a different form, for softening it.

The lips of the player act as a reed to the cornet or trombone. These instruments are open pipes, and, by depressing the valves, the length of the pipe is altered, the wave being made to traverse side channels in addition to its ordinary path.

The air in an open tube may also be thrown into vibration by a burning jet of gas, as in Fig. 198. The pitch depends upon the size of the flame and the length of the tube. By varying the position of the jet in a long tube a series of frequencies in the ratio of 1 : 2 : 3 : 4, &c., can be obtained.

If a gauze diaphragm be inserted in a tube open at both ends, about two inches in diameter and two feet long, at a point three or

four inches from the end, and this diaphragm be heated red hot by a Bunsen burner, upon removing the burner and depressing the tube so as to cause a current of air to pass through it, a very loud note will be produced.

315. Vibrations of Rods and Laminæ.—A plate of metal called a reed is much used for musical purposes in connection with a column of air, as already stated. In parlor organs the sound is produced by the action of vibrating reeds upon air currents, just as in the case of the musical toy called harmonicon. Except in such connection, the sounds of wires and laminæ are generally too feeble to be employed in music. But their vibrations have been much studied, on account of the interesting phenomena attending them.

Such vibrations afford a convenient mode of determining the velocity of sound in solids. A rod, held firmly by a clamp in the middle, and rubbed about half way between the middle and the end by a leather pad well resined, will give a note due to longitudinal vibrations of the rod. While the rod gives out its fundamental note the *ends* vibrate freely, being neither compressed nor extended : but at the centre, held by the clamp, there is no vibration, but a maximum effect of alternate compression when the two pulses meet, and extension when they again travel toward the ends. The middle of the rod is a node. The time of a complete vibration is the time that a pulse would require to travel twice the length of the rod. If the note given is due to 512 vibrations per second, and the length of the rod be x feet, then a length of $2x$ feet is passed over 512 times in a second, and the velocity in the substance of the rod is $512 \times 2x$ feet per second.

316. Wires.—If one end of a steel wire is fastened in a vise and vibrated, while a thin blade of sunlight falls across it, the path of the illuminated point may be traced. It is not ordinarily a circular arc about the fixed point as a centre, but some irregular figure ; and frequently the point describes two systems of ellipses, the vibrations passing alternately from one system to the other several times before running down. If the structure of the wire were the same in every part across its section, and if the fastening pressed equally on every point around it, the orbit of each particle would be a series of ellipses, whose major axes are on the same line. If, moreover, there was no obstruction to the motion, and the law of elasticity could obtain perfectly, it would vibrate in the same elliptic orbit forever, the force toward the centre being directly as the distance. It is easy to cause the wire, in the experiment just described, to vibrate also in parts ; in which case each atom, while describing the elliptic orbit, will perform several

smaller circuits, which appear as waves on the circumference of the larger figure.

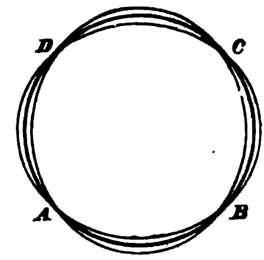
317. Chladni's Plates.—If a square plate of glass or elastic metal, of uniform thickness and density, be fastened by its centre in a horizontal position, and a bow be drawn on its edge, it will emit a pure musical tone; and by varying the action of the bow, and touching different points of the edge with the finger, a variety of sounds may be obtained from it. The plate necessarily vibrates in parts; and the lowest pitch is produced when there are two nodal lines parallel to the sides, and crossing at the centre, thus dividing the plate into four square ventral segments. The position of the nodal lines, and the forms of the segments, are beautifully exhibited by sprinkling writing-sand on the plate. The particles will dance about rapidly till they find the lines of rest, where they will presently be collected. For every new tone the sand will show a new arrangement of nodal lines; and as two or more modes of vibration may coexist in plates, as well as in strings and columns of air, the resultant nodes will also be rendered visible. Again, by fastening the plate at a different point, still other arrangements will take place, each distinguishable by the position of its nodal lines and the pitch of its musical note. The form of the plate itself may also be varied, and each form will be characterized by its own peculiar systems. Chladni, who first performed these interesting experiments, delineated and published the forms of *ninety* different systems of vibration in the square plate alone.

If a fine light powder, as lycopodium (the pollen of a species of fern), be scattered on the plate, it is affected in a very different manner from heavy sand. It will gather into rounded heaps on those portions of the segments which have the greatest amplitude of vibration; the particles which compose the heaps performing a continual circulation, down the sides of the heaps, along the plate to the centre, and up the axis. If the vibration is violent, the heaps will be thrown up from the plate in little clouds over the portions of greatest motion. The cause of this singular effect was ascertained by Faraday, who found that in an exhausted receiver the phenomenon ceased. It is due to a circulation of the air, which lies in contact with a vibrating plate. The air next to those parts which have the greatest amplitude is at each vibration thrown upward more powerfully than elsewhere, and surrounding particles press into its place, and thus a circulation is established; and a fine light powder is more controlled by these atmospheric movements than by the direct action of the plate.

318. Bells.—If a thin plate of metal takes the form of a cylinder or bell, its fundamental note is produced when each ring of the

material changes from a circle to an ellipse, and then into a second ellipse, whose axis is at right angles to that of the former, as seen in Fig. 199. It thus has four ventral segments and four nodal lines, the latter lying in the plane of the axis of the bell or cylinder. If the rings which compose the bell were all detached from one another, they would have different rates of vibration according to their diameter, and hence would produce tones of various pitch; but, being bound together by cohesion, they are compelled to keep the same time, and hence give but one fundamental tone. But a bell, especially if quite thin, may be made to emit a series of harmonic sounds by dividing up into a greater number of segments. It is obvious that the number of nodes must always be *even*, because two successive segments must move in opposite directions in one and the same instant; otherwise the point between them could not be kept at rest, and therefore would not be a node. Besides the principal tone of a church-bell, one or two subordinate sounds on a different pitch may usually be detected. A glass bell, suitably mounted for the lecture-room, will yield *ten* or *twelve* harmonics, by means of a bow drawn on its edge.

FIG. 199.



319. The Voice.—The vocal organ is complex, consisting of a cavity called the *larynx*, and a pair of membranous folds like valves, having a narrow opening between them; this opening, called the *glottis*, admits the air to the larynx from the wind-pipe below. The edges of these valves are thickened into a sort of cord, and for this reason the apparatus is called the *vocal cords*. In the act of breathing, the folds of the glottis lie relaxed and separate from each other, and the air passes freely between them, without producing vibration. But in the effort to form a vocal sound, they approach each other, and become tense, so that the current of air throws them into vibration. These vibrations are enforced by the resonant vibrations of the air of the larynx above; and thus a fullness of sound is produced, as in many musical instruments, in which a reed, and the air of a cavity, perform synchronous vibrations, and emit a much louder sound than either could do alone. If two pieces of thin india-rubber be stretched across the end of a tube, with their edges parallel, and separated by a narrow space, as represented in Fig. 200, the arrangement will give an idea of the larynx and glottis of the vocal

FIG. 200.

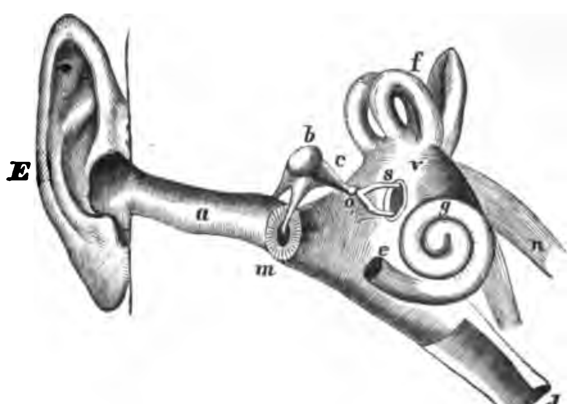
organ. If air be forced through, a sound is produced, whose pitch depends on the size of the tube and the tension of the valves.

The natural key of a person's voice depends on the length and weight of the vocal cords, and the size of the larynx. The yielding nature of all the parts, and the ability, by muscular action, to change the form and size of the cavity and the tension of the valves, give great variety to the pitch, and the power of adjusting it with precision to every shade of sound within certain limits. No instrument of human contrivance can be brought into comparison with the organ of voice. After the voice is formed by its appropriate organ, it undergoes various modifications, by means of the palate, the tongue, the teeth, the lips, and the nose, before it is uttered in the form of articulate speech.

320. The Organ of Hearing.—The principal parts of the ear are the following:

1. The *outer ear*, *E a* (Fig. 201), terminating at the membrane of the tympanum, *m*.

Fig. 201.



2. The *tympanum*, a cavity separated from the outer ear by a membrane, *m*, and containing a series of four very small bones (*ossicles*), *b*, *c*, *o* and *s*, severally called, on account of their form, the *hammer*, the *anvil*, the *ball*, and the *stirrup*. The figure represents the walls of the tympanum as mostly removed, in order to show the internal parts. This cavity is connected with the back part of the mouth by the *Eustachian tube*, *d*.

3. The *labyrinth*, consisting of the *vestibule*, *v*, the *semicircular canals*, *f*, and the *cochlea*, *g*. The latter is a spiral tube, winding two and a half times round. The parts of the labyrinth are excavated in the hardest bone of the body. The figure shows only its exterior. There are two orifices through the bone which sepa-

rates the labyrinth from the tympanum, the round orifice, *e*, passing into the cochlea, and the oval orifice, *s*, leading to the vestibule. These orifices are both closed by a thin membrane. The ossicles of the tympanum form a chain which connects the centre of the membrane, *m*, with that which closes the oval orifice. The labyrinth is filled with a liquid, in various parts of which float the fibres of the auditory nerve.

By the form of the outer ear, the waves are concentrated upon the membrane of the tympanum, thence conveyed through the chain of bones to the membrane of the labyrinth, and by that to the liquid within it, and thus to the auditory nerve, whose fibres lie in the liquid.

CHAPTER IV.

MUSICAL SCALES—THE RELATIONS OF MUSICAL SOUNDS.

321. Numerical Relations of the Notes.—To obtain the series of notes which compose the common scale of music, it is convenient to use the monochord. Calling the sound, which is given by the whole length of the string, the *fundamental*, or *key note*, of the scale, we measure off the following fractions of the whole for the successive notes, namely: $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$. If the whole, and these fractions, are made to vibrate in order, the ear will recognize the sounds as forming the series called the *gamut*, or *diatonic scale*.

Now as the number of vibrations varies inversely as the length of the string, the number of vibrations of these notes respectively, expressed in fractions of the number of vibrations of the whole string, which we will call 1, will be 1, $\frac{9}{8}$, $\frac{5}{4}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{15}{8}$, 2.

If the whole string vibrates 128 times per second, $\frac{8}{9}$ of the string would give $\frac{9}{8}$ of 128 vibrations, or 144 vibrations; $\frac{4}{5}$ of the string would give $\frac{5}{4}$ of 128 vibrations per second, or 160 vibrations. To express the *relative* number of vibrations in the series above, reduce the fractions to a common denominator and compare their numerators, and we have

24, 27, 30, 32, 36, 40, 45, 48.

Sounds whose vibrations per second bear to each other the ratios of the series above are not arbitrarily chosen to form the scale, but they are demanded by the ear. The notes corresponding to the series are named according to their place in the series; thus a note whose vibrations are $\frac{30}{24}$ of the vibrations of the fundamental, is called the third, one whose vibrations are $\frac{36}{24}$ is

the fifth, and that whose vibrations are $\frac{48}{24}$, or twice as many as those of the fundamental, is the *eighth* or octave.

322. Relations of the Intervals.—An *interval* is the relative pitch of two sounds, and its numerical value is expressed by a fraction whose numerator is the number of vibrations per second of the higher sound, and whose denominator is the number of vibrations of the lower or graver sound, or by any fraction equal to this.

In examining the relation of each two successive numbers in the foregoing series, we find three different ratios. Thus,

27 : 24, 36 : 32, and 45 : 40, is each as 9 : 8.
30 : 27 . . . . . and 40 : 36, ... 10 : 9.
32 : 30 . . . . . and 48 : 45, ... 16 : 15.

Therefore, of the seven successive intervals, in the diatonic scale, there are three equal to $\frac{9}{8}$, two equal $\frac{10}{9}$, and two others equal to $\frac{16}{15}$. Each of the first five is called a *tone;* each of the last two is called a *semitone.*

323. Repetition of the Scale.—The eighth note of the scale so much resembles the first in sound, that it is regarded as a repetition of it, and called by the same name. Beginning, therefore, with the half string, where the former series closed, let us consider the sound of that as the fundamental, and take $\frac{8}{9}$ of it for the second, $\frac{4}{5}$ of it for the third, &c.; we then close a second series of notes on the quarter-string, whose sound is also considered a repetition of the former fundamental. Each fraction of the string used in the second scale is obviously half of the corresponding fraction of the whole string, and therefore its note an octave above the note of that. This process may be repeated indefinitely, giving the *second octave, third octave,* &c. Ten or eleven octaves comprehend all sounds appreciable by the human ear; the vibrations of the extreme notes of this entire range have the ratio of $1 : 2^{10}$, or $1 : 2^{11}$; that is, 1 : 1024, or 1 : 2048. Hence, if 16 vibrations per second produce the lowest appreciable note, the highest varies from 16,000 to 33,000. It was ascertained by Dr. Wollaston that the highest limit is different for different ears; so that when one person complains of the piercing shrillness of a sound, another maintains that there is no sound at all. The lowest limit is indefinite for a different reason; the sounds are heard by all, but some will recognize them as low musical tones, while others only perceive a rattling or fluttering noise. Few musical instruments comprehend more than six octaves, and the human voice has only from one to three, the male voice being in pitch an octave lower than the female.

324. Modes of Naming the Notes.—There is one system

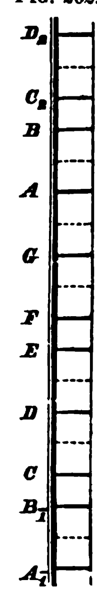
of names for the notes of the scale, which is fixed, and another which is movable. The first is by the seven letters, *A, B, C, D, E, F, G*. The notes of the second octave are expressed by the same letters, in some way distinguished from the former. The best method is to write by the side of the letter the numeral expressing that index of 2, which corresponds to the octave: as A_2, A_3, &c., in the octaves above; A_{I}, A_{II}, in those below.

Any note may be designated as *C*, but this letter is usually assigned to that note which is due to 256 vibrations per second, or middle *C* of the piano-forte.

The second mode of designation is by the syllables, *do, re, mi, fa, sol, la, si*. These express merely the *relations* of notes to each other, *do* always being the fundamental, *re* its second, *mi* its third, &c. In the natural scale, *do* is on the letter *C*, *re* on *D*, &c.; but by the aid of interpolated notes, the scale of syllables may be transferred, so as to begin successively with every letter of the fixed scale.

325. The Chromatic Scale.—Let the notes of the diatonic scale be represented (Fig. 202) by the horizontal lines, *C, D,* &c.; the distance from *C* to *D* being a tone, from *D* to *E* a tone, *E* to *F* a semitone, &c. It will be observed that the fundamental, *C*, is so situated that there are *two* whole tones above it, before a semitone occurs, and then *three* whole tones before the next semitone. *C* is therefore the letter to be called by the syllable *do*, in order to bring the first semitone between the 3d and 4th, and the other semitone between the 7th and 8th, as the figure represents them. Now, that we may be able to transfer the scale of relations to every part of the fixed scale (which is necessary, in order to vary the character of music, without throwing it beyond the reach of the voice), the whole tones are bisected, and two semitone intervals occupy the place of each. The dotted lines in the figure show the places of the interpolated notes, which, with the original notes of the diatonic scale, divide the whole into a series of semitones. This is called the *chromatic* scale. The interpolated note between *C* and *D* is written *C*♯ (*C* sharp), or *D*♭ (*D* flat), and so of the others. As the whole tones lie in groups of *twos* and *threes*, so the new notes inserted are grouped in the same way. This explains the arrangement of the *black keys* by twos and threes alternately in the key-board of the organ and piano-forte. The white keys compose the diatonic scale, the white and black keys together, the chromatic scale. It is obvious

Fig. 202.



that on the chromatic scale any one of the twelve notes which compose it may become *do*, or the fundamental note, since the required series, 2 tones, 1 semitone, 3 tones, 1 semitone, can be arranged to succeed each other, at whatever note we begin the reckoning. This change, by which the fundamental note is made to fall on different letters, is called the *transposition* of the scale.

326. Chords and Discords.—When two or more sounds, meeting the ear at once, form a combination which is agreeable, it is called a *chord;* if disagreeable, a discord. The disagreeable quality of a discord, if attended to, will be perceived to consist in a certain roughness or harshness, however smooth and pure the simple sounds which are combined. On examining the combinations, it will be found that if the vibrations of two sounds are in some very simple relations, as 1 : 2, 1 : 3, 2 : 3, 3 : 4, &c., they produce a chord; and the lower the terms of the ratio, the more perfect the chord. On the other hand, if the numbers necessary to express the relations of the sounds are large, as 8 : 9, or 15 : 16, a discord is produced. It appears that concordant sounds have *frequent* coincidences of vibrations. If, in two sounds, there is coincidence at every vibration of each, then the pitch is the same, and the combination is called *unison.* If every vibration of one coincides with every alternate vibration of the other, the ratio is 1 : 2, and the chord is the *octave*, the most perfect possible. The *fifth* is the next most perfect chord, where every second vibration of the lower meets every third of the higher, 2 : 3. The *fourth*, 3 : 4, the *major third*, 4 : 5, the *minor third*, 5 : 6, and the *sixth*, 3 : 5, are reckoned among chords; while the *second*, 8 : 9, and the *seventh*, 8 : 15, are harsh discords. What is called the *common chord* consists of the 1st, 3d, and 5th, combined, and is far more used in music than any other. *Harmony* consists of a succession of chords, or rather, of such a succession of combined sounds as is pleasing to the ear; for discords are employed in musical composition, their use being limited by special rules. Many combinations, which would be too disagreeable for the ear to dwell upon, or to finish a musical period, are yet quite necessary to produce the best effect; and without the relief which they give, perfect harmony, if long continued, would satiate.

327. Temperament.—This is a term applied to the small errors introduced into the notes, in tuning an instrument of fixed keys, in order to adapt the notes equally to the several scales. If the tones were all equal, and if semitones were truly half tones, no such adjustment of notes would be needed; they would all be exactly correct for every scale. Representing the notes in

the scale whose fundamental is C by the numbers in Art. 321, we have,

$$\begin{array}{cccccccccccc} C, & D, & E, & F, & G, & A, & B & C_2, & D_2, & E_2, & \&c. \\ 24, & 27, & 30, & 32, & 36, & 40, & 45, & 48, & 54, & 60, & \&c. \end{array}$$

Now suppose we wish to make D, instead of C, our key-note; then it is obvious that E will not be exactly correct for the second on the new scale. For the fundamental to its second is as 8 to 9; and 8 : 9 :: 27 : 30.375, instead of 30. Therefore, if D is the key-note, we must have a new E, slightly above the E of the original scale. So we find that A, represented by 40, will not serve to be the 5th in the new scale; since 2 : 3 :: 27 : 40.5, which is a little higher than A (= 40). After adding these and other new notes, to render the intervals all exactly right for the new key of D, if we proceed in the same manner, and make E (= 30) our key-note, and obtain its second, third, &c., exactly, we shall find some of them differing a little, both from those of the key of C, and also of the key of D. Using in this way all the twelve notes of the chromatic scale in succession for the fundamental, it appears that several different E's, F's, G's, &c., are required, in order to make each scale perfect. In instruments, whose sounds cannot be modified by the performer, like the organ and piano-forte, as it is considered impossible to insert all the pipes or strings necessary to render every scale perfect, such an *adjustment* is made as to distribute these errors equally among all the scales. For example, E is not made a perfect *third* for the key of C, lest it should be too *imperfect* for a *second* in the key of D, and for its appropriate place in other scales. It is this equal distribution of errors among the several scales which is called *temperament*. The errors, when thus distributed, are too small to be observed by most persons; whereas, if an instrument was tuned perfectly for any one scale, all others would be intolerable.

The word temperament, as above explained, has no application except to instruments of fixed keys, as the organ and piano-forte; for, where the performer can control and modify the notes as he is playing, he can make every key perfect, and then there are no errors to be distributed. The flute-player can roll the flute slightly, and thus humor the sound, so as to cause the same fingering to give a precisely correct *second* for one scale, a correct *third* for another, and so on. The player on the violin does the same, by touching the string in points slightly different. The organs of the voice, especially, can be adjusted to make the intervals perfect on *every* scale. In these cases there is no *tempering*, or dividing of errors among different scales, but a *perfect adjustment* to each scale, by which all error is avoided.

328. Harmonics.—The fact has been mentioned that a string, or a column of air, may vibrate in parts, even while vibrating as a whole. It only remains to show the musical relations of the sounds thus produced. When a string vibrates in parts, it divides into halves, thirds, fourths, or other *aliquot* parts. Now, a half-string produces an *octave* above the whole, making the most perfect chord with it. The third of a string being two-thirds of the half-string, produces the *fifth* above the octave, a very perfect chord. The quarter-string gives the *second octave;* the fifth part of it, being $\frac{4}{5}$ of the quarter, gives the *major third* above the second octave; and the sixth part, being $\frac{4}{6}$ of the quarter, gives the *fifth* above the second octave. Thus, all the simpler divisions, which are the ones most likely to occur, are such as produce the best chords; and it is for this reason that the sounds are called *harmonics*. The same is true of air-columns and bells. The Æolian harp furnishes a beautiful example of the harmonics of a string. Two or more fine smooth cords are fastened upon a box, and tuned, at suitable intervals, like the strings of a violin; and the box is placed in a narrow opening, where a current of air passes. Each string at different times, according to the intensity of the breeze, will emit a pure musical note; and, with every change, will divide itself in a new mode, and give another pitch, while it will frequently happen that the vibrations of different divisions will coexist, and their harmonic sounds mingle with each other.

329. Overtones.—But the parts into which a sounding body divides do not always harmonize with the whole. For instance, $\frac{1}{7}$ or $\frac{1}{11}$ of a string is discordant with the fundamental. The word *harmonics* is not, therefore, applicable except to a very few of the many possible sounds which a body may produce. The word *overtone* is used to express in general any sound whatever, given by a part of a sounding body. A string may furnish 20 or 30 overtones, but only a small number of them would be harmonics.

The presence of these overtones may be determined by means of the resonator devised by Helmholtz and modified by König, which can be adjusted to respond to a great variety of notes; if on drawing out the cylinder a tone is produced, it must be that the same tone exists in the compound sound under investigation.

330. Timbre, or Quality of Tone.—Even when the pitch of two sounding bodies is the same, the ear almost always distinguishes one sound from the other by certain qualities of tone peculiar to each. Thus, if the same letter be sounded by a flute

and the string of a piano, each note is easily distinguished from the other. Two church-bells may be upon the same key, and yet one be agreeable, and the other harsh to the ear.

As a result of his researches Helmholtz decides that the timbre is determined by the overtones which accompany the primary tones. If these overtones could be eliminated, leaving only the pure, simple tones, a note of given pitch sounded by a flute would not be distinguishable from a note of the same pitch sounded by a violin.

A long monochord can, by varying the mode of exciting the vibrations, be made to yield a great variety of sounds, while there is perceived in them all the same fundamental undertone which determines the pitch. If the string be struck at the *middle*, then no node can be formed at that point; hence, the mixed sound will contain no overtones of the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, or other *even* aliquot parts of the string; for all such would require a node at the middle. But if struck at *one-third* of its length from the end, then the overtones, $\frac{1}{2}$, $\frac{1}{4}$, &c., may exist, but not those of $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$, or any other parts whose node would fall at $\frac{1}{3}$ of the length from the end.

For reasons which are mostly unknown, some sounding bodies have their fundamental accompanied by harmonic overtones, and others by overtones which are discordant. And this is one cause of the agreeable, or unpleasant, quality of the sounds of different bodies.

331. Communication of Vibrations.—The acoustic vibrations of one body are readily communicated to others, which are near or in contact. We have already noticed that the vibrations of a reed will excite those of a column of air in a pipe. If two strings, which are adapted to vibrate alike, are fastened on the same box, and one of them is made to sound, the other will sound also more or less loudly, according to the intimacy of their connection. The vibrations are communicated partly through the air, and partly through the materials of the box. So, if a loud sound is uttered near a piano-forte, several strings will be thrown into vibration, whose notes are heard after the voice ceases. The noticeable fact in all such experiments is, that the vibrations thus communicated from one body to another cause sounds which *harmonize* with each other, and with the original sound. For the rate of vibration will either be identical, or have those simple relations which are expressed by the smallest numbers. Let a person hold a pneumatic receiver or a large tumbler before him, and utter at the mouth of it several sounds of different pitch; and he will probably find some one pitch which will be distinctly rein-

forced by the vessel. That particular note, which the receiver by its size and form is adapted to produce, will not be called forth by a sound that would be discordant with it. The melodeon, seraphine, and instruments of like character, owe their full and brilliant notes to reeds, each of which has its cavity of air adapted to vibrate in unison with it. It sometimes happens that the second body, vibrating as a whole, would not harmonize with the first, and yet will give the same note by some mode of division. Thus it is that all the various sounds of the monochord, and of the strings of the viol, are reinforced by the case of thin wood upon which they are stretched. The plates of wood divide by nodal lines into some new arrangement of ventral segments for every new sound emitted by the string. In like manner, the pitch of the tuning-fork, and all the rapid notes of a music-box, are rendered loud and full by the table, in contact with which they are brought. The extended material of the table is capable of division into a great variety of forms, and will always give a sound in unison with the instrument which touches it.

332. One System of Vibrations Controlling Another.—If two sounding bodies are nearly, but not precisely on the same key, they will sometimes, when brought into close contact, be made to harmonize perfectly. The vibrations of the more powerful will be communicated to the other, and control its movements so that the discordance, which they produce when a few inches apart, will cease, and concord will ensue. Two diapason pipes of an organ, tuned a quarter-tone or even a semitone from unison, so as to jar disagreeably upon the ear, when one inch or more asunder, will be in perfect unison, if they are in contact through their whole length. Even the slow oscillations of two watches will influence each other; if one gains on the other only a few beats in an hour, then, if they are placed side by side on the same board, they will beat precisely together.

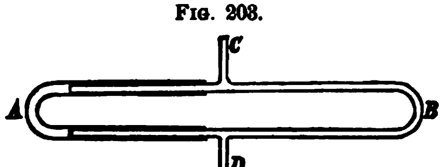
333. Crispations of Fluids.—Among the numerous acoustic experiments illustrating the communication of vibrations, none are more beautiful than those in which the vibrations of glass rods are conveyed to the surface of a fluid. Let a very shallow pan of glass or metal be attached to the middle of a thin bar of wood, three or four feet long, and resting near its ends on two fixed bridges; let water be placed in the pan, and a long glass rod standing in it, or on the wood, be vibrated longitudinally, by drawing the moistened fingers down upon it; the liquid immediately shows that the vibrations are communicated to it. The surface is covered with a regular arrangement of heaps, called *crispations*, which vary in size with the pitch of sound, which is produced

by the same vibration. If the pitch is higher, they are smaller, and may be readily varied from three or four inches in diameter to the fineness of the teeth of a file. Crispations of the same character are also formed in clusters on the water in a large tumbler or glass receiver, when the finger is drawn along its edge; every ventral segment of the glass produces a group of hillocks by the side of it on the surface of the water.

334. Interference of Waves of Sound.—Whenever two sounds are moving through the air, every particle will, at a given instant, have a motion which is the resultant of the two motions which it would have had if the sounds were separate. These motions may conspire, or they may oppose each other. The word *interference* is used in scientific language to express the *resultant effect*, whatever it may be.

If two sound waves of equal intensity and of the same length move together so that any phase of one is coincident with the like phase of the other the resultant sound has greater intensity than either of the components; if, however, any phase of one is coincident with the opposite phase of the other, entire extinction of the sound results, and we have silence. To illustrate this experimentally, take two pieces of tubing and bend them into the form shown in Fig. 203, the branch A being large enough to slide over the other, and insert a whistle at D. The sound waves travel to the ear placed at C by two different routes, starting in the same phase at the point D. If the branches A and B are of equal length, or differ by one or more whole wave lengths, the waves will meet at C in the same phase and produce a sound of greater intensity than that of either alone; but if the branch A be drawn out or pushed in till the route A differs in length from B by half a wave length, or by any odd multiple of half a wave length, opposite phases will meet at C and destroy each other, producing silence. A much more simple experiment to show the same effect is the following: The two prongs of a tuning-fork always vibrate in opposite directions, one producing a condensation in the direction in which the other produces rarefaction, thus destroying each other's effect by interference, and hence the almost total absence of sound when the fork is held free in the hand. In such case, if the sound waves from one prong be intercepted by slipping over it without contact a paper cylinder, the

FIG. 203.



sound is augmented. Hold a vibrating fork so that either the back face of a prong, or the side faces of the two prongs, are parallel to the ear, and the sound will be distinctly audible; turn the fork about its axis 45° from either of these positions, and silence results. During one entire rotation of the fork there will be four positions of maximum intensity and four other positions of total extinction of the sound. If the fork be rotated over the mouth of a resonating jar, the effect is much more striking.

The *beats*, which are frequently heard in listening to two sounds, indicate the points of maximum condensation produced by the union of the condensed parts of both systems of waves. And the sounds are considered discordant when these beats are just so frequent as to produce a disagreeable fluttering or rattling. If too near or too far apart for this, they are regarded practically as concordant. And when the beats are too close to be perceived separately, yet the peculiar adjustment of condensations of one system with those of the other, according as *one* wave measures *two*, or *two* waves measure *three*, or *four* measure *five*, &c., is at once distinguished by the ear, and recognized as the chord of the *octave*, the *fifth*, the *third*, &c. When a sound and its octave are advancing together, there are instants in which any given particle of air is impressed with two opposite motions, and other alternate moments when both motions are in the same direction. For the waves of the highest sound are half as long as those of the lowest; hence, while every *second* condensation of the former coincides with *every* condensation of the latter, the alternate ones of the former must be at the points of greatest rarefaction of the latter; and this cannot occur without opposite movements of the particles.

335. Number and Length of Waves for Each Note.— Though the vibrations of any musical note are too rapid to be counted, yet the number may be ascertained in several ways.

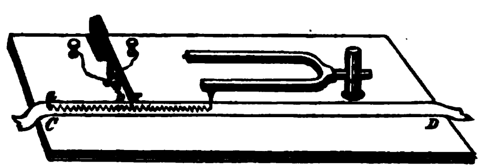
If an elastic slip of metal be clamped at one end so that the other end may rest against a toothed wheel, and the wheel be revolved with different velocities, musical notes of different pitch will be produced. If a uniform velocity be maintained for a given time, sixty seconds for instance, and the number of revolutions of the toothed wheel be read from an indicator suitably connected with the wheel, then the number of teeth upon the wheel multiplied by the number of revolutions gives the number of impulses communicated to the air in the given time; this product, divided by the seconds in the time, sixty in the case supposed, gives the number of vibrations per second. The instrument is called Savart's Wheel.

Another method of determining the number of vibrations is by means of the *siren*, invented by De La Tour. In this instrument the pulses are produced by puffs of air, in rapid succession, caused by revolving a disc, perforated around its circumference by numerous holes which pass in front of an air jet. The calculation is similar to that given for Savart's wheel.

In the method given above it is difficult in practice to maintain a constant velocity. A graphic representation of the vibrations, devised by Duhamel, is without this objection. Without giving details of the construction, the principle of the method may be given as follows:

Support the vibrating rod—a tuning-fork for example—above a table, as in Fig. 204; to one prong of the fork attach a fine steel point or style, which shall very lightly touch the strip of paper *D C*, which has been coated with a film of lamp black; place at *I* an electric style, which is in connection with a pendulum beating seconds, which shall make a dot upon the paper at each beat of the pendulum. If the paper be moved in the direction *D C* while the fork is kept vibrating, an undulating line will be traced upon the film, and the number of undulations between any two consecutive *second's* dots, as *b a*, gives the number of vibrations of the fork per second.

FIG. 204.



In these ways it is ascertained that the numbers corresponding to the letters of the scale are the following:

$$\begin{cases} C, & D, & E, & F, & G, & A, & B, & C_2, \\ 128, & 144, & 160, & 170\frac{2}{3}, & 192, & 213\frac{1}{3}, & 240, & 256, \end{cases}$$

The highest note of the above series, C_2, 256, is the middle *C* of the piano-forte.

There is not, however, a perfect agreement of pitch in different countries, and among different classes of musicians. Accordingly, *C*, which is given above as corresponding to 128 vibrations per second, has several values, varying from 127 to 131.

336. Doppler's Principle.—Thus far the hearer and the source of sound have been supposed not to change their relative positions. When a sounding body approaches the ear, the tone perceived is higher than that due to the number of vibrations per second, since more vibrations per second reach the ear than if the body had remained at rest with respect to the hearer. Suppose the sound to be middle *C*, and the sounding body and the hearer

to remain relatively stationary, then 256 vibrations per second will be communicated to the ear: If now the sounding body approach at the rate of 66 feet per second there will be perceived, in addition to the 256 vibrations, $\frac{66}{4.4} = 15$ vibrations per second, and the pitch will be that due to 271 vibrations per second instead of 256. If the sounding body recede from the hearer the opposite effect will be produced.

337. Acoustic Vibrations Visibly Projected.—The vibrations of heavy tuning-forks can be magnified and rendered distinctly visible to an audience by projecting them on a screen. The fork being constructed with a small metallic mirror attached near the end of one prong, a sunbeam reflected from the mirror will exhibit all the movements of the fork greatly enlarged on a distant wall; and if the fork is turned on its axis, the luminous projection will take the form of a waving line. And by the use of two forks, all the phenomena of interference may be rendered as distinct to the eye as they are to the ear.

338. Edison's Phonograph.—This is an instrument for recording sounds and for reproducing them at any subsequent time. In the modern instrument the records are taken upon cylinders of wax. A cylinder is rotated around its axis at a uniform speed. At every revolution the axis is displaced in its own direction by a distance equal to the distance between two contiguous threads of a screw, this screw being cut upon one end of the shaft upon which the cylinder turns. The sound to be recorded is received upon a diaphragm, whose plane is parallel to the axis of the cylinder. The sound makes the diaphragm vibrate, and these vibrations cause a cutting style, which is attached to the rear of the diaphragm, to cut in the wax a spiral undulating record of the sound. To reproduce the sound the cutting style is replaced by a smooth, pointed style. This is placed at the beginning of the spiral record, and the cylinder is again revolved at its former speed. The style, following the undulations, causes the diaphragm to vibrate in the same manner as when it was subjected to the original sound vibrations. An ear placed over the diaphragm also receives these as if they were coming from the original source.

The phonograph is capable of recording the three characteristics of tones—pitch, loudness, and quality. The nearer together the indentations are, the higher the pitch; the deeper they are, the greater the intensity; and the more irregular the outline of the individual indentations, the richer in overtones is the sound recorded.

PART V.

OPTICS.

CHAPTER I.

MOTION AND INTENSITY OF LIGHT.

339. Definitions.—Light is supposed to consist of exceedingly minute and rapid vibrations in a medium or ether which fills space; which vibrations, on reaching the retina of the eye, cause vision, as the vibrations of the air cause hearing, when they impinge on the tympanum of the ear, and as thermal vibrations produce a sensation of warmth, when they fall on the skin; the difference between light and heat is solely a difference in wave length, waves longer than those of the extreme red, or shorter than the extreme violet, producing no effect upon the optic nerve.

Bodies, which of themselves are able to produce vibrations in the ether surrounding them, are said to *emit* light, and are called *self-luminous*, or simply *luminous;* those, which only *reflect* light, are called *non-luminous.* Most bodies are of the latter class. A *ray* of light is a line, along which light is propagated; a *beam* is made up of many parallel rays; a *pencil* is composed of rays either diverging or converging; and is not unfrequently applied to those which are parallel.

A substance, through which light is transmitted, is called a *medium;* if objects are clearly seen through the medium, it is called *transparent;* if seen faintly, *semi-transparent;* if light is discerned through a medium, but not the objects from which it comes, it is called *translucent;* substances which transmit no light are called *opaque,* though all are more or less translucent when cut in sufficiently thin laminæ.

340. Light Moves in Straight Lines.—So long as the medium continues uniform, the line of each ray is perfectly

straight. For an object cannot be seen through a bent tube; and if three discs have each a small aperture through it, a ray cannot pass through the three, except when they are exactly in a straight line. The shadow which is projected through space from an opaque body proves the same thing; for the edges of the shadow, taken in the direction of the rays, are all straight lines.

From every point of a luminous surface light emanates in all possible directions, when not prevented by the interposition of an opaque body. Thus, a candle is seen by night at the distance of one or two miles; and within that limit, no space so small as the pupil of the eye is destitute of rays from the candle. A point from which light emanates is called a *radiant*. If light from a radiant falls perpendicularly on a circular disk, the pencil is a cone; if on a square disk, it is a square pyramid, &c., the illuminated surface in each case being the base, and the radiant the vertex.

341. The Velocity of Light.—It has been ascertained by several independent methods, that light moves at the rate of about 186,300 *miles per second.*

One method is by means of the eclipses of Jupiter's satellites. The planet Jupiter is attended by four moons which revolve about it in short periods. These small bodies are observed, by the telescope, to undergo frequent eclipses by falling into the shadow which the planet casts in a direction opposite to the sun. The exact moment when the satellite passes into the shadow, or comes out of it, is calculated by astronomers. But sometimes the earth and Jupiter are on the same side, and sometimes on opposite sides of the sun; consequently, the earth is, in the former case, the whole diameter of its orbit, or about one hundred and eighty-five millions of miles nearer to Jupiter than in the latter. Now it is found by observation, that an eclipse of one of the satellites is seen about sixteen minutes and a half sooner when the earth is nearest to Jupiter, than when it is most remote from it, and consequently, the light must occupy this time in passing through the diameter of the earth's orbit, and must therefore travel at the rate of about 186,868 miles per second, according to this determination.

Another method of estimating the velocity of light, wholly independent of the preceding, is derived from what is called the *aberration of the fixed stars.* The apparent place of a fixed star is altered by the motion of its light being combined with the motion of the earth in its orbit. The place of a luminous object is determined by the direction in which its light meets the eye. But the direction of the impulse of light on the eye is modified by the

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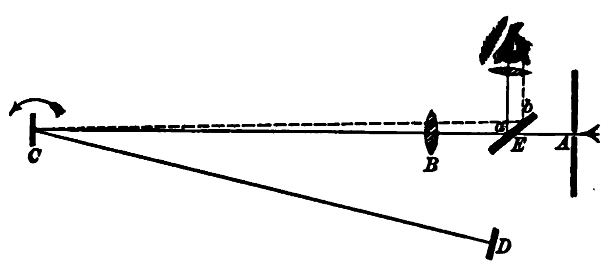
motion of the observer himself, and the object appears forward of its true place. The stars, for this reason, appear slightly displaced in the direction in which the earth is moving; and the velocity of the earth being known, that of light may be computed in the same manner as we determine one component, when the angles and the other component are known.

342. Determination of the Velocity of Light by Experiment.—The velocity of light has been determined also by experiment, in a manner somewhat analogous to that employed by Wheatstone for ascertaining the velocity of electricity. The method adopted by Foucault is essentially the following:

Through an aperture *A*, in a shutter (Fig. 205), a beam of light is admitted, which passing through an inclined transparent

FIG. 205.



glass mirror, *E*, and through a lens of very long focus, *B*, falls upon a mirror *C*, and is reflected to a mirror *D*; the mirror *D* again reflects the beam back to *C*, whence it is returned through the lens *B* to the glass mirror *E*, is reflected, and finally enters the eye. The mirror *C* is so mounted as to rotate with great velocity upon an axis, perpendicular to the plane of the paper in the case supposed.

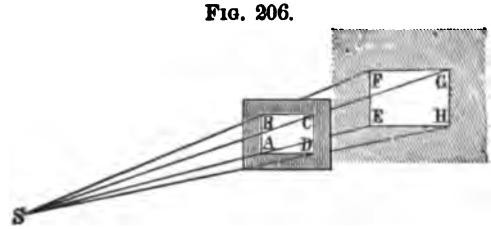
If the mirror *C* rotate slowly, in the direction of the arrow, the beam will alternately disappear and reappear at the point *a*; but if the velocity be increased to 30 or more revolutions per second the impression on the eye becomes persistent and the beam is seen without interruptions, appearing stationary at *a*. If now the speed of the mirror *C* be increased to from 300 to 600 revolutions per second, the change of position of the mirror *C*, while light is passing from it to *D* and back again, is sufficient to return the reflected beam to some point *b*, the distance from *a* depending upon the velocity of rotation. From the displacement at *b*, the angular motion of the mirror at *C*, while the beam traverses the distance from *C* to *D* and back again, can be determined; and knowing the rate of rotation, this fraction of one turn gives

the time which the light required to traverse double the distance $C\,D$, and hence its velocity. Such is an outline of the mode of experimenting, all details for securing precision having been omitted.

By this method the velocity given in Art. 341—186,300 miles per second—was determined by A. A. Michelson, U. S. N. The distance between the revolving mirror C and the mirror D was 2000 feet.

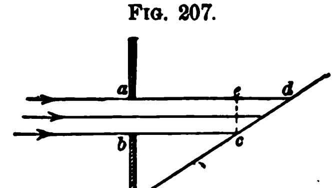
343. Loss of Intensity by Distance.—The *intensity* of light varies *inversely as the square of the distance.* In Fig. 206, suppose light to radiate from S, through the rectangle $A\ C$, and fall on $E\,G$, parallel to $A\ C$. As $S\,A\,E$, $S\ B\ F$, &c., are straight lines, the triangles, $S\ A\ B$, $S\ E\ F$, are similar, as also the rectangles, $A\ C$, $E\ G$; therefore, $A\ C : E\ G :: A\ B^2 : E\,F^2 :: S\,A^2 : S\,E^2$. But the same quantity of light, being diffused over $A\ C$ and $E\ G$, will be more intense, as the surface is smaller. Hence, the intensity of light at E : intensity at $A :: A\ C : E\ G :: S\,A^2 : S\,E^2$, which proves the proposition. This demonstration is applicable to every kind of emanation in straight lines from a point.

FIG. 206.



If the surface receiving the light be oblique to the axis of the beam, the intensity of illumination is proportional to the sine of the angle which the rays make with the surface. Let Fig. 207 represent a section through the axis of a beam passing through an orifice $a\ b$ and falling upon the inclined surface at $c\ d$. Now because of the obliquity, the surface $c\ d$ is greater than the section at right angles to the beam represented by $e\ c$, and hence is less intensely illuminated at any point. But surface $e\ c$: surface $c\,d$ :: line $e\,c$: line $c\ d$ :: sine $e\,d\,c$: sine $c\,e\,d$.

FIG. 207.



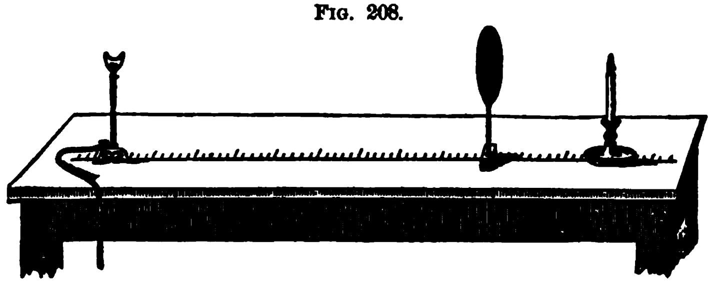
Calling the illumination upon any point of the right section $e\,c$ unity or u, the illumination upon any point of $c\ d$ will be $u \times \frac{e\,c}{c\,d} = u \times$ sine $e\,d\,c$.

344. Brightness the Same at all Distances.—The *bright-*

ness of an object is the quantity of light which it sheds, as compared with the apparent area from which it comes. Now the *quantity* (or intensity), as has just been shown, varies inversely as the square of the distance. The apparent *area* of a given surface also diminishes in the same ratio, as we recede from it. Hence the brightness is constant. For illustration, if we remove to *three* times the distance from a luminous body, we receive into the eye nine times less light, but the body also appears nine times smaller, so that the relation of light to apparent area remains the same.

345. Bunsen's Photometer.—Photometers are instruments for determining the relative intensities of two sources of light. The Bunsen (Fig. 208) photometer makes use of the fact, that a grease spot, upon a piece of bibulous white paper, appears darker than the surrounding paper, if it be more intensely illuminated on the side toward the observer; and appears lighter than the paper, if more intensely illuminated on the side away from the observer. When equally illuminated on both sides, the spot is invisible. The two lights to be compared are placed on opposite sides of the

FIG. 208.



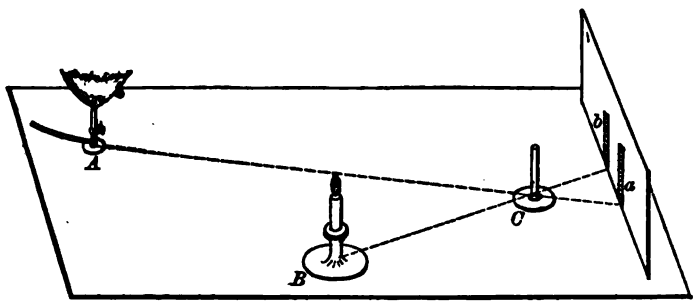
grease-spot screen. One of them is then moved until the spot becomes invisible, then the intensities of the two lights are as the squares of their distances from the spot. For, suppose that one of the lights is a standard candle, and that it is placed at a unit's distance from the screen. Any other light, at the same distance, illuminating the screen with the same intensity, would have an intensity of one candle power. If, at the same distance, the illumination were 16 times as intense, then this light has an intensity of 16 candle power. Now, the eye cannot estimate the value of different intensities. By the grease spot, however, it can tell when two intensities are equal. Suppose that a candle at unit's distance is balanced by an incandescent lamp at 4 units' distance. From Art. 343 we know that the lamp would illuminate the screen with 16 times the intensity if it were moved up to a unit's distance. It is, then, a 16 ($=4^2$) candle-power lamp.

The distance of the standard light from the screen may always

be considered as unity. Then the intensity of the other is equal to the square of its distance (measured in this unit) from the screen.

346. Rumford's Photometer.—Let the two unequal lights be placed at A and B (Fig. 209) so that the shadows of an opaque rod C shall fall side by side upon a screen, as at a and b. The portion of the screen upon which the shadow a falls receives light only from the candle B and none from the gas flame A; the portion b is illuminated by A alone. The opaque body thus secures for each light a portion of the screen which it alone illuminates.

FIG. 209.



Now move either light towards or from the screen until the two portions a and b are equally illuminated by their respective lights, and then measure the distances from A to b, $= m$, and from B to a, $= n$. B at distance n illuminates the screen as intensely as A at distance m.

Then, as in Art. 345, $A : B = m^2 : n^2$;

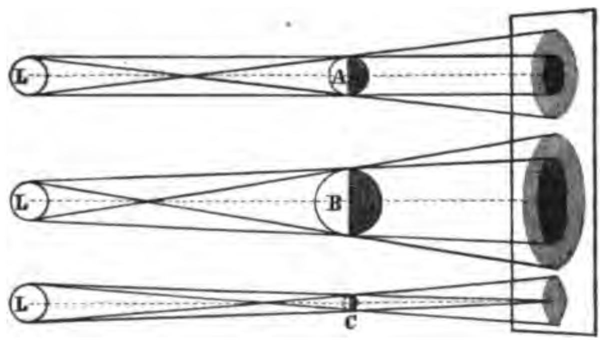
or, *the intensities vary directly as the squares of those distances from the screens at which equal illumination is obtained.*

347. Shadows.—When a luminous body shines on one which is opaque, the space beyond the latter, from which the light is excluded, is called a *shadow*. The same word, as commonly used, denotes only the *section* of a shadow made by a surface which crosses it. Shadows are either *total* or *partial*. If tangents are drawn on all the corresponding sides of the two bodies, the space inclosed by them beyond the opaque body is the total shadow; if other tangents are drawn, crossing each other between the bodies, the space between the total shadow and the latter system of tangents is the partial shadow, or *penumbra*. In case the bodies are spheres, as in Fig. 210, the total shadow will be a cylinder, or conical frustum, each of infinite length, or a complete cone, according to the relative size of the spheres. But, in every case, the penumbra and inclosed total shadow will form an increasing frustum. It is obvious that the shade of the penumbra grows

gradually deeper from the outer surface to the total shadow within it.

Every shadow cast by the sun has a penumbra bordering it, which gives to the shadow an ill-defined edge; and the more

FIG. 210.



remote the sectional shadow is from the opaque body which casts it, the broader will be the partial shadow on the edge.

If instead of a luminous body of sensible magnitude, the source of light be a point, then no penumbra will be formed. The electric arc between carbon points casts a sharply-defined shadow of a hair upon a screen placed at a great distance.

CHAPTER II.

REFLECTION OF LIGHT.

348. Radiant and Specular Reflection.—Light is said to be *reflected* when, on meeting a surface, it is turned back into the same medium. In ordinary cases of reflection, the light is diffused in all directions, and it is by means of the light thus scattered from a body that it becomes visible, when it sheds no light of its own. This is called *radiant reflection.* It is produced by unpolished surfaces. But when a surface is highly polished, a beam of light falling on it is reflected in some particular direction; and, if the eye is placed in this reflected beam, it is not the reflecting surface which is seen, but the original object, apparently in a new position. This is called *specular reflection.* It is, however, generally accompanied by some degree of radiant reflection, since the reflector itself is commonly visible in all directions. Ordinary

mirrors are not suitable for accurate experiments on reflection, because light is modified by the glass through which it passes. The *speculum* is therefore used, which is a reflector made of solid metal, and accurately ground to any required form, either *plane*, *convex*, or *concave*. The word *mirror* is, however, much used in optics for every kind of reflector.

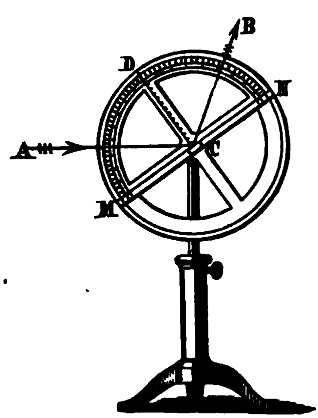
Optical experiments are usually performed on a beam of light admitted through an aperture into a darkened room; the direction of the beam being regulated by an adjustable mirror placed outside. An instrument consisting of a plane speculum moved by a clock, in such a manner that the reflected sunbeam shall remain stationary at all hours of the day, is called a *heliostat*.

349. The Law of Reflection.—When a ray of light is incident on a mirror, the angle between it and a perpendicular to the surface at the point of incidence, is called the *angle of incidence;* and the angle between the reflected ray and the same perpendicular, is called the *angle of reflection*. The law of reflection found to be universally true is the following:

The incident ray, the reflected ray, and the normal to the surface are in the same plane, and the normal bisects the angle which these rays make with each other.

This is well shown by attaching a small mirror to the centre of a graduated semicircle perpendicular to its plane. Let *M D N* (Fig. 211) be the semicircle, graduated from *D* both ways to *M* and *N*, and mounted so that it can be revolved on its centre, and clamped in any position. Let the small mirror be at *C*, with its plane perpendicular to *C D*; then a ray from the heliostat, as *A C*, passing the edge at a particular degree, will be seen after reflection to pass the corresponding degree in the other quadrant. By revolving the semicircle, any angle of incidence may be tried, and the two rays are always found to be in the same plane with *C D*, and equally inclined to it.

FIG. 211.



As the mirror revolves, the reflected ray revolves twice as fast.

For *A C D* is increased or diminished by the angle through which the mirror turns; therefore *D C B* is also increased or diminished by the same; hence *A C B*, the angle between the two

rays, is increased or diminished by the sum of both, or twice the same angle.

It follows from the law of reflection, that a ray which falls on a mirror perpendicularly, retraces its own path after reflection. It is obvious, also, that the complements of the angles of incidence and reflection are equal, i. e. $A\,C\,M = B\,C\,N$. The law of reflection is applicable to curved as well as to plane mirrors; the radius of curvature at any point being the perpendicular with which the incident and reflected rays make equal angles.

Radiant reflection forms no exception to the foregoing law, though the incident rays are in one and the same direction, and the reflected rays are scattered every way. For the minute cavities and prominences which constitute the roughness of the general surface are bounded by small surfaces lying at all inclinations; and each one reflecting the rays which meet it in accordance with the law, those rays are necessarily thrown off in all possible directions.

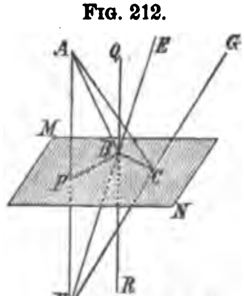
The proportion of the incident light reflected varies with the angle of incidence. When light strikes the surface of water perpendicularly only .018 is reflected, the rest entering the water; but at an incidence of $89\frac{1}{2}°$ .721 is reflected. In the case of mercury .666 is reflected at perpendicular incidence, while .721 is reflected at $89\frac{1}{2}°$, the non-reflected rays entering the metal and being destroyed, as light.

350. Inclination of Rays to each other not altered by the Plane Mirror:—

1. Rays which *diverge* before reflection, diverge at the same angle after reflection.

Let $M\,N$ (Fig. 212) be a plane mirror, and $A\,B$, $A\,C$, any two rays of light falling upon it from the radiant A, and reflected in the lines $B\,E$, $C\,G$. Draw the perpendicular $A\,P$, and produce it indefinitely, as to F, behind the mirror; also produce the reflected rays back of the mirror. Let $Q\,R$ be perpendicular to the mirror at the point B; it is therefore parallel to $A\,F$, and the plane passing through $A\,F$ and $Q\,R$, is that which includes the ray $A\,B$, $B\,E$. Therefore $E\,B$, when produced back of the mirror, intersects $A\,P$ produced. Let F be the point of intersection. $B\,A\,F = A\,B\,Q$, and $A\,F\,B = E\,B\,Q$; but $A\,B\,Q = E\,B\,Q$ (Art. 349); $\therefore B\,A\,F = A\,F\,B$, and $A\,B = F\,B$. If P and B be joined, $P\,B$ being in

FIG. 212.



the plane $M N$ is perpendicular to $A F$, and therefore bisects it. Hence, the reflected ray meets the perpendicular $A F$ as far behind the mirror, as the incident ray does in front. In the same way it may be proved that $A C = C F$, and that $C G$, when produced back of the mirror, meets $A F$ at the same point F.

Now, since the triangles $A C B$ and $F C B$, have their sides respectively equal, their angles are equal also; hence $B A C = B F C$. Therefore any two rays diverge at the same angle after reflection as they did before reflection.

Since the reflected rays seem to emanate from F, that point is called the *apparent radiant*; A is the *real radiant*.

2. Rays which *converge* before reflection, converge at the same angle after reflection. Let $E B$, $G C$, be incident rays converging toward F, and let $B A$, $C A$, be the reflected rays. It may be proved as before, that A and F are in the same perpendicular, $A F$, and equidistant from P, and that $E F G = B A C$.

The point F, to which the incident rays were converging, is called the *virtual focus*; A is the *real focus*.

3. Rays which are *parallel* before reflection are parallel after reflection.

It has been proved in case 1, that F, the intersection of the reflected rays, is as far behind the mirror, as A, the intersection of incident rays, is before it. Now, if the incident rays are parallel, A is at an infinite distance from the mirror. Therefore F is at an infinite distance behind it, and the reflected rays are parallel.

In all cases, therefore, rays reflected by a plane mirror retain the same inclination to each other which they had before reflection.

351. Spherical Mirrors.—A *spherical mirror* is one which forms a part of the surface of a sphere, and is either convex or concave. The *axis* of such a mirror is that radius of the sphere which passes through the middle of the mirror. In the practical use of spherical mirrors, it is found that the light must strike the surface very nearly at right angles; hence, in the following statements, the mirror is supposed to be a very small part of the whole spherical surface, and the rays nearly coincident with the axis.

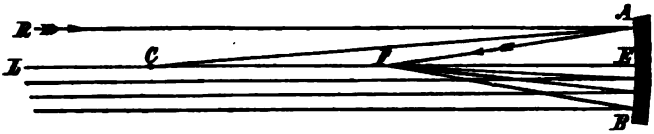
It is sufficient to trace the course of the rays on one side of the axis, since, on account of the symmetry of the mirror around the axis, the same effect is produced on every side.

352. Converging Effect of a Concave Mirror.—

1. *Parallel* rays are converged to the *middle* point between the centre and surface, which is therefore called the *focus of parallel rays* or the *principal focus*. Let $R A$, $L E$ (Fig. 213), be parallel

rays incident upon the concave mirror $A\ B$, whose centre of concavity is C. The ray $L\ E$, passing through C, and therefore

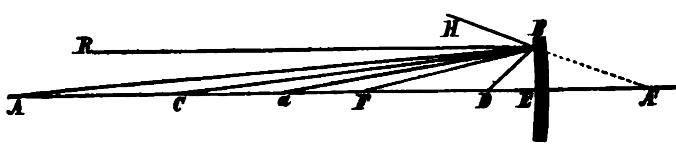
Fig. 213.



perpendicular to the mirror at E, is reflected directly back. Join $C\ A$, and make $C\ A\ F = R\ A\ C$; then $R\ A$ is reflected in the line $A\ F$, and the two reflected rays meet at F. $R\ A\ C = A\ C\ F$, $\therefore A\ C\ F = F\ A\ C$, and $A\ F = C\ F$, and as A and E are very near together, $E\ F = A\ F = F\ C$; that is, the focus of parallel rays is at the middle point between C and E.

2. *Diverging* rays, falling on a given concave mirror, are reflected *converging, parallel, or less diverging*, according to the degree of divergency in the original pencil. Let C (Fig. 214) be the centre of concavity, and F the focus of parallel rays. Then,

Fig. 214.



rays diverging from any point, A, beyond C, will be converged to some point, a, between C and F, since the angles of incidence and reflection are less than those for parallel rays. Rays diverging from C are reflected back to C; those from points between C and F, as a, are converged to points beyond C, as A; those diverging from F become parallel; and those from points between F and the mirror, as D, *diverge* after reflection, but at a less angle than before, and seem to flow from A'. To prove, in the last case, that the angle of divergence, A', after reflection, is less than the angle D, the divergence before reflection, observe that the angle A' is less than the exterior angle $H\ B\ C$, or its equal, $D\ B\ C$ (Art. 349); and $D\ B\ C$ is less than the exterior, $A'\ D\ B$; much more, then, is A' less than $A'\ D\ B$.

3. *Converging* rays are made to *converge more*. The rays $H\ B$, $A\ E$, converging to A', are reflected to D, nearer the mirror than F is. And it has been shown that the angle D is larger than A', hence the convergency is increased.

From the three foregoing cases, it appears that the *concave*

mirror always tends to produce *convergency;* since, when it does not actually produce it, it diminishes divergency.

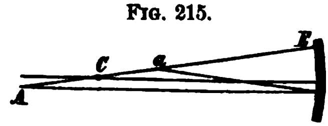
The principal focus can be determined practically by receiving the sun's rays upon the mirror, parallel to its axis, and finding the point at which a sharp image of the sun is formed. The distance of this image from the surface is one-half the radius of curvature.

353. Conjugate Foci.—When light radiates from A, it is reflected to a (Fig. 214); when from a, it meets at A. Any two such interchangeable points are called *conjugate foci.* If the radius of the mirror and the distance of one focus from the mirror are given, the distance of its conjugate focus may be determined. Let the radius $= r$; the distance $A\ E = m$; and $a\ E = n$. As the angle $A\ B\ a$ is bisected by $B\ C$, $A\ B : a\ B :: A\ C : a\ C$; that is, since $B\ E$ is very small, $A\ E : a\ E :: A\ C : a\ C$, or, $m : n :: m - r : r - n$.

$$\therefore m = \frac{n\ r}{2\ n - r};\ \text{and}\ n = \frac{m\ r}{2\ m - r}.$$

If A is not on the axis of the mirror, as in Fig. 215, let a line be drawn through A and C, meeting the mirror in E; this is called a *secondary* axis, and the light radiating from A will be reflected to a on the same secondary axis, for $A\ E$ is perpendicular to the mirror, and will be reflected directly back; and if $A\ E$ and $C\ E$ are given, $a\ E$ may be found as before.

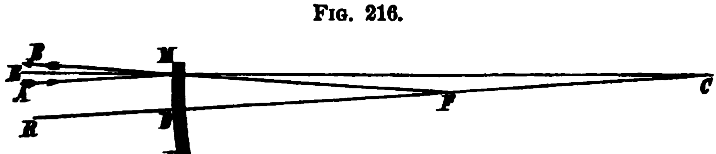
FIG. 215.



354. Diverging Effect of a Convex Mirror.—

1. *Parallel* rays are reflected diverging from the *middle point* between the centre and surface. Let C (Fig. 216) be the centre of convexity of the mirror $M\ N$, and draw the radii, $C\ M$, $C\ D$,

FIG. 216.

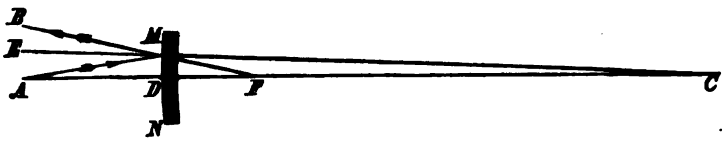


producing them in front of the mirror; these are perpendicular to the surface. The ray $R\ D$ will be reflected back; $A\ M$ will be reflected in $M\ B$, making $B\ M\ E = A\ M\ E$. Produce the reflected ray back of the mirror, and it will meet the axis in F, midway from C to D; for $F\ C\ M = A\ M\ E$, and $F\ M\ C = B\ M\ E$; therefore the triangle $F\ C\ M$ is isosceles, and $C\ F = F\ M$, and as

M is very near D, $C F = F D$. Hence the rays, after reflection, diverge as if they radiated from a point in the middle of $C D$, which is the apparent radiant.

2. *Diverging* rays have their divergency increased. Let $A D$, $A M$ (Fig. 217), be the diverging rays; $D A$, $M B$, the reflected

FIG. 217.



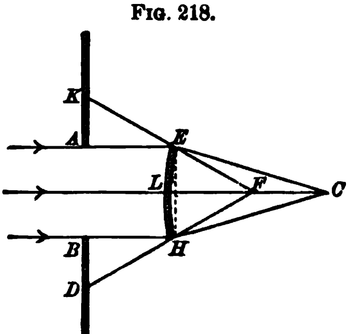
rays; these when produced meet at F, which is the apparent radiant. $M A F$ is the divergency of the incident rays, and $A F B$ of the reflected rays. Now the exterior angle, $A F B$, is greater than $C M F$, or $B M E$, or $A M E$. But $A M E$, being exterior, is greater than $M A F$; much more, then, is $A F B$ greater than $M A F$.

3. *Convergent* rays are at least rendered less convergent, and may become parallel or divergent, according to the degree of previous convergency. The two first effects are shown by Figs. 216 and 217, reversing the order of the rays. And it is easy to perceive that rays converging to C, will diverge from C after reflection; if to a point more distant than C, they will diverge afterward from a point between C and F (Fig. 216), and *vice versa*.

The general effect, therefore, of a *convex* mirror, is to produce *divergency*.

A and F (Fig. 217) are called conjugate foci, being interchangeable points; for rays from A move after reflection as though from F, and rays converging to F are by reflection converged to A. Conjugate foci, in the case of the convex mirror, are in the same axis either principal or secondary, as they are in the concave mirror, and for the same reason, viz., that every axis is perpendicular to the surface.

FIG. 218.



Their relative positions may be determined by the formula, easily deduced, as in Art. 353,

$$m = \frac{n r}{2 n + r}.$$

To determine the radius of curvature experimentally: Through a circular opening in a screen whose diameter is greater than $E H$ (Fig. 218), receive the sun's rays upon the mirror, parallel to the axis, and move the screen so

that the diameter $K D$ of the illuminated circle is twice the chord $E H$ of the mirror; then measure

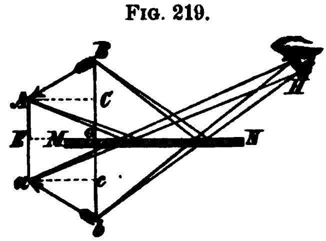
$$D H = F H = F L = \tfrac{1}{4} \text{ Rad.}$$

To render this method more accurate cover all of the mirror, except a small central circle, with some opaque covering, and use only the exposed portion as above.

355. Images by Reflection.—An optical image consists of a collection of focal points, from which light either really or apparently radiates. When rays are converged to a focus they do not stop, but cross, and diverge again, as if originally emanating from the focal point. A collection of such points, arranged in order, constitutes a *real image.* When rays are reflected diverging, they proceed *as though* they emanated from a point behind the mirror. A collection of such imaginary radiants forms an *apparent* or *virtual image.* The images formed by *plane* and *convex* mirrors are always apparent; those formed by *concave* mirrors may be of either kind.

356. Images by a Plane Mirror.—When an object is before a plane mirror, its image is at the *same distance behind* it, of the *same magnitude,* and *equally inclined* to it. Let $M N$ (Fig. 219) be a plane mirror, and $A B$ an object before it, and let the position of the object be such that the reflected rays may enter the eye placed at H. From A and B let fall upon the plane of the mirror the perpendiculars $A E$, $B G$, and produce them, making $E a = A E$, and $G b = B G$. Now, since the rays from A will, after reflection, radiate as if from a (Art. 350), and those from B, as if from b, and the same of all other points, therefore the image and object are equally distant from the mirror. $A C$, $a c$, parallel to the mirror, are equal; as $B G = b G$, and $A E = a E$, therefore, by subtraction, $B C = b c$; also the right angles at C and c are equal. Therefore $A B = a b$, and $B A C = b a c$; that is, the object and image are of equal size, and equally inclined to the mirror.

FIG. 219.



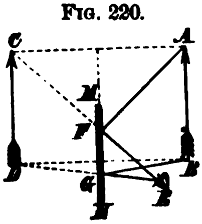
It appears from the demonstration, that the object and its image are comprehended between the same perpendiculars to the plane of the mirror; and this image will appear in the same position whatever may be the position of the eye.

The object and image obviously have to each other twice the inclination that each has to the mirror. Hence, in a mirror inclined 45° to the horizon, a horizontal surface appears vertical, and one which is vertical appears horizontal.

357. Symmetry of Object and Image.—All the three dimensions of the object and image are respectively *equal*, as shown above, but one of them is *inverted* in position, namely, that dimension which is perpendicular to the mirror. Hence, a person and his image face in opposite directions; and trees seen in a lake have their tops downward. Those dimensions which are parallel to the mirror are not inverted. In consequence of the inversion of *one* dimension alone, the object and its image are not *similar*, but *symmetrical* forms; and one could not coincide with the other if brought to occupy the same space. The image of a *right* hand is a *left* hand, and all relations of right and left are reversed. It is for this reason that a printed page, seen in a mirror, is like the type with which it was printed.

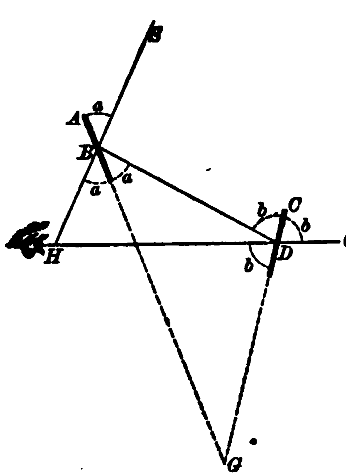
358. The Length of Mirror Requisite for Seeing an Object.—If an object is parallel to a mirror, the length of mirror occupied by the image is to the length of the object as the reflected ray to the sum of the incident and reflected rays. Let $A\ B$ (Fig. 220) be the length of the object, $C\,D$ that of the image, and $F\,G$ that of the space occupied on the mirror; then, by similar triangles, $F\,G : C\,D :: E\,F : E\,C$. But $C\,D = A\ B$, and $C\,F = A\ F$; $\therefore\ F\,G : A\ B :: E\ F : A\ F + F\ E$. If the eye is brought nearer the mirror, the space on the mirror occupied by the image is diminished, because $E\,F$ has to $A\ F + F\ E$ a less ratio than before. The same effect is produced by removing the object further from the mirror. The length of mirror necessary for a person to see himself is equal to half his height, because in that case, $E\ F : A\ F + F\ E :: 1 : 2$, which ratio will not be altered by change of distance.

FIG. 220.



359. Displacement of Image by Two Reflections.—If an image is seen by light reflected from two mirrors in a plane perpendicular to their common section, its angular deviation from the object is equal to twice the inclination of the mirrors. Let $A\ B$, $C\,D$ (Fig. 221) be two plane

FIG. 221.



mirrors inclined at the angle $A\ G\ C$. If an eye at H sees the star S in the direction O, the angle $S\ H\ O = 2\ A\ G\ C$.

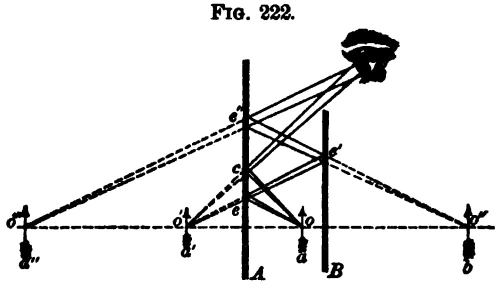
For the exterior angle $C\ D\ B = b = a + G$, or $2\ b = 2\ a + 2\ G$, and $B\ D\ O = 2\ b = 2\ a + H$; hence $2\ a + H = 2\ a + 2\ G$; therefore $H = 2\ G$.

This principle is employed in the construction of *Hadley's quadrant*, and the *sextant*, used at sea for measuring angular distances. The angles measured are twice as great as the arc passed over by the index which carries the revolving mirror; hence, in the quadrant, an arc of 45° is graduated into 90°; and, in the sextant, an arc of 60° is graduated into 120°.

360. Multiplied Images by Two Mirrors.—

1. *Parallel Mirrors.* The series of images is *infinite* in number, and arranged *in a straight line*, perpendicular to the mirrors. The object a, between the parallel mirrors, A and B (Fig. 222), has an image at a', as far behind A as a is in front of it. To

FIG. 222.



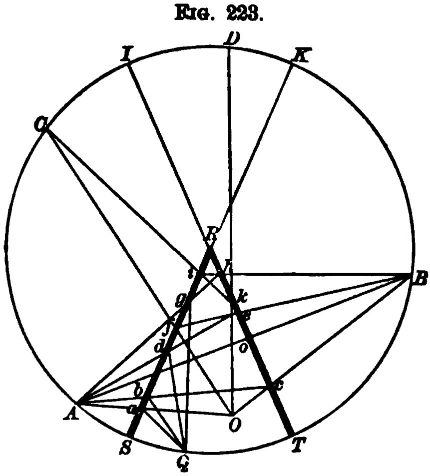
avoid confusion, a pencil from only one point o is drawn, once reflected at c, and entering the eye as though it came from o'. The rays reflected by A diverge as though they emanated from a'; hence, the light reflected from A upon B may be regarded as proceeding from a *real* object at a', whose image will be b, as far back of B as a' is in front of B. The light reflected from B to A again diverges as though it really came from b, and regarding b as a *real* object as before *its* image would be formed at a'' as far behind A as b is in front of it. The pencil which enters the eye seems to proceed from o''', having been reflected from e'', as though it came from o'', its reflection in this case having been from e' as though it came from o', though it was really reflected from e after having emanated from o. The pencil which would enter the eye from a third image at the left of a'' may be traced through all its reflections in like manner. As light is absorbed and scattered at each reflection the number of such images is limited.

The multiplied images of a small bright object, sometimes

seen in a looking-glass, are produced by repeated reflections between the front and the silvered covering on the back side. At each internal impact on the first surface some light escapes, and shows us an image, while another portion is reflected to the back, and thence forward again. The image of a lamp viewed very obliquely in a mirror is sometimes repeated eight or ten times; and a planet, or bright star, when seen in a looking-glass, will be accompanied by three or four faint images, caused in the same way.

2. *Inclined Mirrors.* Let Q (Fig. 223) be the object, and O the position of the eye. With R as a centre and radius $R\,Q$, describe a circumference.

Fig. 223.



Suppose a chord $Q\,A$ to be drawn perpendicular to the mirror S, then A will be the image of Q. Regarding A as a *real* object, as in the case of parallel mirrors, draw a chord $A\,B$ perpendicular to the mirror T, then since $o\,B = o\,A$, B will be the image of A. Suppose a chord $B\,C$ to be drawn perpendicular to the mirror S, then C, being as far behind the mirror S as the object B, assumed as *real*, is in front of it, will be the image of B; and for like reasons D will be the image of C in mirror T. All the images formed by the inclined mirrors are thus seen to be confined to the circumference of a circle described as above stated. There can be no image of D, since it lies behind both mirrors prolonged. The image A is seen by rays which proceed from Q to a and thence to the eye at O. B is seen as though the rays came from B to O, these having been reflected at c as though they came from A, the reflection at b being direct from Q. The image C is seen by rays reflected from the points f, e, d; and D by rays reflected from k, i, h, g. The reflections occur in the order d, e, f, and g, h, i, k. Only images formed by light first reflected from S have been considered; a second series produced by light first reflected from T may be constructed in like manner.

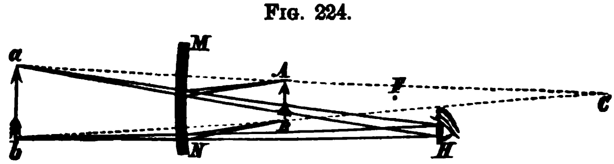
361. The Kaleidoscope.—This instrument, when carefully constructed, beautifully exhibits the phenomenon of multiplied

reflection by inclined mirrors. It consists of a tube containing two long, narrow, metallic mirrors, inclined at a suitable angle; and is used by placing the objects (fragments of colored glass, &c.) at one end, and applying the eye to the other. In order that there may be perfect symmetry in the figure made up of the objects and their successive images, the angle of the mirrors should be of such size, that it can be exactly contained an even number of times in 360°. The best inclination is 30°; and the field of view is then composed of 12 sectors. It is also essential, that the small objects forming the picture, should lie at the least possible distance beyond the mirrors. To insert three mirrors instead of two, as is often done, only serves to confuse the picture, and mar its beauty.

362. Images by the Concave Mirror.—The concave mirror forms various images, either *real* or *apparent*, either *greater* or *less* than the object, either *erect* or *inverted*, according to the place of the object.

1. The object *between the mirror and its principal focus.* By Art. 352 (2), rays which diverge from a point between the mirror and its principal focus, continue to diverge after reflection, but in a less degree. Let C be the centre, and F the principal focus of the mirror $M\ N$ (Fig. 224), and $A\ B$ the object. Draw the axes, $C\ A$, $C\ B$, and produce them behind the mirror. The pencil from

FIG. 224.



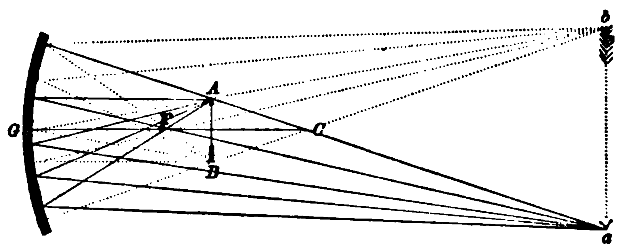
A will be reflected to the eye at H, radiating as from a, in the same axis; likewise, that from B, as from b. Therefore, the image is *apparent*, since rays do not actually flow from it; *erect*, as the axes do not cross each other between the object and image; *enlarged*, because it subtends the angle of the axes at a greater distance than the object does. As the object approaches, and finally reaches the principal focus, the reflected rays approach parallelism, and the image departs from the mirror, till it is at an infinite distance.

Other rays than those given in the figure fall upon the mirror from A, but are reflected either above or below the eye, and therefore have no part in the production of the image, and for that reason are omitted. The same is true of rays from every other point of the object.

2. Object *between the principal focus and the centre.* As soon as the object passes the principal focus, the rays of each pencil begin to converge; and each radiant of the object has its conjugate focus in the same axis beyond the centre (Art. 353).

For example the rays diverging from the point A, represented in Fig. 225 by full lines, after reflection are converged to a situated somewhere on the secondary axis $A\ C\ a$, and rays from B,

FIG. 225.



given as dotted lines, converge finally to b on the axis $B\ C\ b$. The images of intermediate points are formed in the same way.

If an observer is beyond $a\ b$, the rays, after crossing at the image, will reach him, as though they originated in $a\ b$; or if a screen is placed at $a\ b$, the light which is collected in the focal points will be thrown in all directions by radiant reflection from the screen. Hence, the image is *real;* it is also *inverted,* because the axes cross between the conjugate foci; and it is *enlarged,* since it subtends the angle of the axes at a greater distance than the object does. That $b\ C$ is greater than $B\ C$, is proved by joining $C\ G$, which bisects the angle $B\ G\ b$, and therefore divides $B\ b$ so that $B\ C : C\ b :: B\ G : G\ b$. As $G\ b$ is greater than $B\ G$, so $C\ b$ is greater than $B\ C$. When the object reaches the centre, the image is there also, but inverted in position, since rays which proceed from one side of C, are reflected to the other side of it.

3. Object *beyond the centre.* This is the reverse of (2), the conjugate foci having changed places; $a\ b$, therefore, being the object, $A\ B$ is its image, *real, inverted, diminished.* As the object removes to infinity, the image proceeds only to the principal focus F.

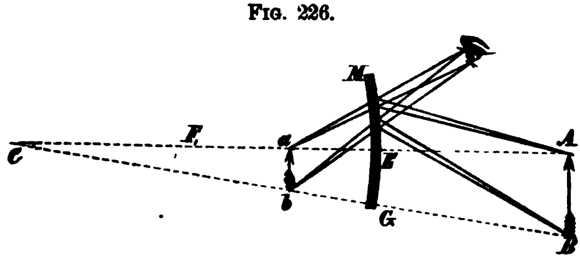
363. Illustrated by Experiment.—These cases are shown experimentally by placing a lamp close to the mirror, and then carrying it along the axis to a considerable distance away. While the lamp moves from the mirror to the principal focus, its image behind the mirror recedes from its surface to infinity; we may then regard it as being either at an infinite distance behind, or an infinite distance in front, since the rays of every pencil are par-

allel. After the lamp passes the principal focus, the image appears in the air at a great distance in front, and of great size, and they both reach the centre together, where they pass each other; and, as the lamp is carried to great distances, the image, growing less and less, approaches the principal focus, and is there reduced to its smallest size. The only part of the infinite line of the axis before and behind, in which no image can appear, is the small distance between the mirror and its principal focus.

If a person looks at *himself*, so long as he is between the mirror and the principal focus, he sees his image behind the mirror and enlarged. But when he is between the principal focus and centre, the image is *real*, and behind him; the converging rays of the pencils, however, enter his eyes, and give an indistinct view of his image as if at the mirror. When he reaches the centre, the pupil of the eye is seen covering the entire mirror, because rays from the centre are perpendicular, and return to it from all parts of the surface. Beyond the centre, he sees the real image in the air before him, distinct and inverted.

364. Images by the Convex Mirror.—The convex mirror affords no variety of cases, because diverging rays, which fall upon it, are made to diverge still more by reflection. In Fig. 226 the pencil from A is reflected, as if radiating from a in the same

FIG. 226.

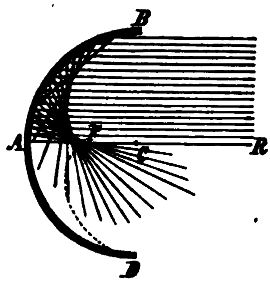


axis $A\ C$, and that from B, as from b in the axis $B\ C$; and there apparent radiants are always nearer the surface than the middle point between it and C (Art. 354). The image is therefore *apparent;* it is *erect*, since the axes do not cross between the object and image; and it is *diminished*, as it subtends the angle of the axes at a less distance than the object.

As in Fig. 224, rays from A and B, which after reflection pass above or below the eye, have been omitted; only that portion of the mirror from which rays are represented as being reflected has any part in the formation of the image. For eyes in other positions other rays would be used, still seeming to come from the same image $a\ b$.

365. Caustics by Reflection.—These are luminous curved surfaces, formed by the intersections of rays reflected from a hemispherical concave mirror. The name *caustic* is given from the circumstance that *heat*, as well as light, is concentrated in the focal points which compose it. *B A D* (Fig. 227), represents a section of the mirror, and *B F D* of the caustic; the point *F*, where all the sections of the caustic through the axis meet each other, is called the *cusp*. When the incident rays are parallel, as in the figure, the cusp is at the principal focus, that is, the middle point between *A* and *C*. The rays near the axis *R A*, after reflection meet at the cusp (Art. 352); but those a little more distant cross them, and meet the axis a little further toward *A*. And the more distant the incident ray from the axis, the further from the centre does the reflected ray meet the axis. Thus each ray intersects all the previous ones, and this series of intersections constitutes the curve, *B F*. The curve is luminous, because it consists of the foci of the successive pencils reflected from the arc *A B*.

Fig. 227.



If the incident rays, instead of being parallel, diverge from a lamp near by, the form of the caustic is a little altered, and the cusp is nearer the centre. This case may be seen on the surface of milk, the light of the lamp being reflected by the edge of the bowl which contains it.

If parallel or divergent light falls on a convex hemispherical mirror, there will be *apparent* caustics behind the mirror; that is, the light will be reflected as if it radiated from points arranged in such curves.

366. Spherical Aberration of Mirrors.—It has already been mentioned (Art. 351), that the statements in this chapter relating to focal points and images, as produced by spherical mirrors, are true only when the mirror is a very small part of the whole spherical surface. In Art. 365 we have seen the effect of using a large part of the spherical surface—viz., the rays neither converge *to*, nor diverge *from* a single point, but a series of points arranged in a curve. This general effect is called the *spherical aberration* of a mirror; since the deviation of the rays is due to the spherical curvature. The deviation, as we have seen, is quite apparent in a hemisphere, or any considerable portion of one; but it exists in some degree in any spherical mirror, unless infinitely small compared with the hemisphere.

But there are curves which will reflect without aberration. Let a concave mirror be ground to the form of a paraboloid, and rays parallel to its axis will be converged to the focus without aberration. For, at any point on such a mirror, a line parallel to the axis, and a line drawn to the focus, make equal angles with the tangent, and therefore, equal angles with the perpendicular to the surface. And rays, parallel to the axis of a convex paraboloid, will diverge as if from its focus, on the same account. Again, if a radiant is placed at the focus of a concave parabolic mirror, the reflected rays will be parallel to the axis, and will illuminate at a great distance in that direction. Such a mirror, with a lamp in its focus, is placed in front of the locomotive engine to light the track, and has been much used in light-houses. If a concave mirror is ellipsoidal, light emanating from one focus is collected without aberration to the other, because lines from the foci to any point of the curve make equal angles with the tangent at that point.

Since heat is reflected according to the same law as light, a concave mirror is a burning-glass. When it faces the sun, the light and heat are both collected in a small image of the sun at the principal focus. And, if no heat were lost by the reflection, the intensity at the focus would be to that of the direct rays, as the area of the mirror to the area of the sun's image. Burning mirrors have sometimes been constructed on a large scale, by giving a concave arrangement to a great number of plane mirrors.

CHAPTER III.

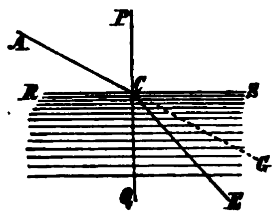
REFRACTION OF LIGHT.

367. Division of the Incident Beam.—When light falls on an *opaque* body, we have noticed that it is arrested, and a shadow formed beyond. Of the light thus arrested, a portion is reflected, and another portion lost, which is said to be absorbed by the body. When light meets a *transparent* body, a part is still reflected, and a small portion absorbed, but, in general, the greater part is transmitted. The ratio of intensities in the reflected and transmitted beams varies with the angle of incidence, but little being reflected at small angles of incidence, and almost the whole at angles near 90°.

368. Refraction.—The transmitted beam suffers important changes, one of which is a change in *direction*. This change is

called *refraction*, and takes place at the surface of a new medium. In Fig. 228, *A C*, incident upon *R S*, the surface of a different medium, is turned at *C* into another line, as *C E*, which is called the *refracted ray*. The angle *E C Q*, between the refracted ray and the perpendicular is called the *angle of refraction*; the angle *G C E*, between the directions of the incident and the refracted rays, is the *angle of deviation*.

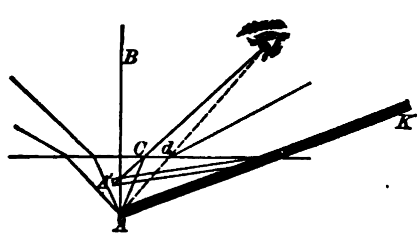
Fig. 228.



It is a general fact, to which there are but few exceptions, that a ray of light in passing out of a rarer into a denser medium is refracted *toward* the perpendicular to the surface; and in passing out of a denser into a rarer medium, it is refracted *from* the perpendicular. But the chemical constitution of bodies sometimes affects their refracting power. Some inflammable bodies, as sulphur, amber, and certain oils, have a great refracting power in comparison with other bodies; and in a given instance, a ray of light in passing out of one of these substances into another of greater density may be turned from the perpendicular instead of toward it. In the optical use of the words, therefore, *denser* is understood to mean, *of greater refractive power*; and *rarer* signifies, *of less refractive power*. In Fig. 228, the medium below *R S* is of greater refractive power than that above.

Let *A K* (Fig. 229) represent a straight rod, the lower end *A* being beneath the surface of water. The rays which diverge from the point *A* are bent from the perpendicular *A B*. The ray *A d*, which, if prolonged, would enter the eye, is by refraction bent so as to pass below, while the ray *A C* deviates at *C* and enters the eye as though coming from *A'*, thus giving to the rod the bent appearance noticed in an oar when in use.

Fig. 229.



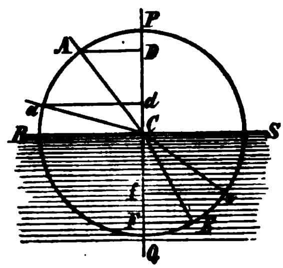
In the same manner, the bottom of a river appears elevated, and diminishes the apparent depth of the stream. Let a small object be placed in the bottom of a bowl, and let the eye be withdrawn till the object is hidden from view by the edge of the bowl. If now the bowl be filled up with water, the object is no longer concealed, for the light, as it emerges from the water, is bent away from the perpendicular, and brought low enough to enter the eye.

369. Law of Refraction.—The law which is found to hold true in all cases of common refraction is this:

The angles of incidence and refraction are on opposite sides of the perpendicular to the surface, and, for any given media, the sines of the angles have a constant ratio for all inclinations.

For example, in Fig. 230, if $A\ C$ is refracted to E, then $a\ C$ will be refracted to e, so that $A\ D : E\ F :: a\ d : e\ f$; and if the rays pass out in a contrary direction, the ratio is also constant, being the reciprocal of the former, viz., $E\ F : A\ D :: e\ f : a\ d$.

Fig. 230.



This constant ratio is called the *Index of Refraction* and is found by dividing the sine of the angle of incidence by the sine of the angle of refraction.

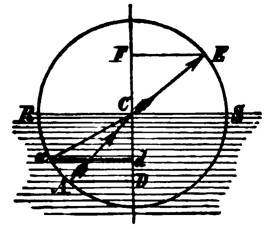
A ray perpendicular to the surface, passing in either direction, is not refracted; for, according to the law, if the sine of one angle is zero, the sine of the other must be zero also.

The following table gives the indices of refraction, the ray being supposed to pass from a vacuum into the substance; such indices are termed *absolute indices*:

Diamond	2.450	Crown glass (mean)	1.580
Carbon disulphide	1.678	Alcohol	1.372
Oil of cassia	1.630	Water	1.336
Flint glass (mean)	1.600	Ice	1.309
Quartz	1.548	Air	1.000294
Canada Balsam	1.540		

370. Limit of Transmission from a Denser to a Rarer Medium.—As a consequence of the law of refraction, there is a limit beyond which a ray cannot escape from a denser medium. Let $A\ C$ (Fig. 231) be the ray incident upon the rarer medium $R\ E\ S$. It will be refracted from the perpendicular $D\ F$ into the direction $C\ E$, so that $A\ D$ is to $E\ F$ in a constant ratio (Art. 369). If the angle $A\ C\ D$ be increased, $F\ C\ E$ must also increase till at length its sine equals $C\ S$.

Fig. 231.



Suppose the denser medium to be water and the rarer air, then

$$\frac{\text{Sine } A\ C\ D}{\text{Sine } E\ C\ F} = \frac{1}{1.336} \text{ nearly;}$$

hence, sine $E C F = 1.336 \times$ sine $A C D$. If $E C F$ be increased to 90°, then sine $90° = 1 = 1.336 \times$ sine $A C D$, from which we find sine $A C D = .7485$, the angle corresponding to which is 48° 28′. If the angle of incidence be greater than 48° 28′, its sine would exceed .7485, and therefore the sine of the angle in air should exceed unity, which is impossible. Hence it follows, that whenever the angle of incidence is greater than that at which the sine of the angle of refraction becomes equal to radius, the ray cannot be refracted consistently with the constant ratio of the sines.

This is proved also by experiment; the emerging ray increases its angle of refraction till it at length ceases to pass out. Beyond that limit all the incident rays are *reflected* from the inner surface of the denser medium; and this reflection is more perfect than any external reflection, and is called *total reflection.*

The limiting angle for diamond is 24° 12′, and its great brilliancy, when properly cut, is due to numerous internal total reflections which cause the light to emerge in different directions.

371. Opacity of Mixed Transparent Media.—Light in passing from a medium to a different one, is partly reflected and partly refracted; if this be often repeated in a mixed medium no light is transmitted. It is the frequency of reflection at the limiting surfaces of air and water that renders foam opaque. So also a transparent crystal, when crushed, becomes an opaque powder. If the powder be wetted with a liquid having the same refractive index as the crystal, the reflections will be prevented and transparency will result.

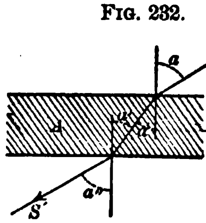
372. Transmission through Parallel Plane Surfaces.—Let S (Fig. 232) enter the medium A, and represent the emergent ray by S'. Suppose the ray to enter from a vacuum, and to emerge into a vacuum again, and call the index of refraction m. Then

$$\text{sine } a = m \times \text{sine } a', \text{ and}$$

$$\text{sine } a' = \frac{1}{m} \text{ sine } a'';$$

multiplying these together, we have sine $a =$ sine a'', whence $a = a''$, and the emergent and incident rays are parallel. Suppose the ray S' to enter a second medium B, bounded by parallel faces, it will emerge parallel to S', and therefore parallel to S. Hence if a ray traverse any number of media with parallel faces, these media being separated by vacua the

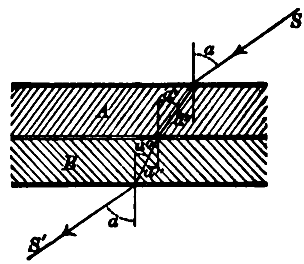
FIG. 232.



finally emergent ray will be parallel to the first incident ray S. If now the spaces between the media be diminished, the result will not be changed, and finally when the diminution reaches its limit the faces of the media will be in contact, and we shall still have the incident and emergent rays parallel.

373. Determination of Relative Indices of Refraction.—When a ray passes from a medium A into another B (Fig. 233), the absolute indices of these being known the relative index may be found. Suppose the media to be bounded by parallel plane faces. Let m be the *absolute index* of A, and n that of B. Denote the relative index, $\frac{\text{sine } a'}{\text{sine } a''}$, by i. Suppose the ray S to enter A from a vacuum, then

FIG. 233.



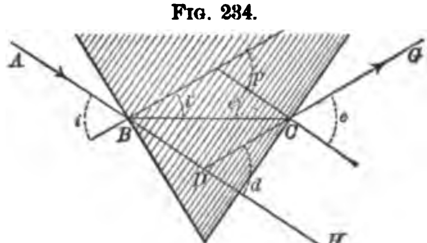
$$\text{sine } a = m \times \text{sine } a'$$
$$\text{sine } a' = i \times \text{sine } a''$$
$$\text{sine } a'' = \frac{1}{n} \text{ sine } a$$ since the emergent ray S' is parallel to the incident ray S (Art. 372). By multiplying these equations together, we find $i = \frac{n}{m}$; hence, to find the relative index of refraction *when a ray passes from medium A into medium B, divide the absolute index of B by that of A.*

Suppose a ray to pass from air into carbon disulphide, then $\frac{1.678}{1.0003} = 1.6774$, knowing which the deviation of the ray for any given angle of incidence can be found.

The same principle may be applied to find the relative index of two substances whose relative indices with respect to a third are known.

374. Transmission through a Medium Bounded by Inclined Planes.—A medium bounded by *inclined* planes is called a *prism*. The angle included by the planes through which the light passes is called the *refracting angle* of the prism, and the planes are *deviating planes*.

FIG. 234.



Let A B (Fig. 234) be the incident ray, and C G

the emergent ray. The total deviation will be $G D H = d$. Adopting the notation of the figure, we have $G D H = D B C + D C B$ or $d = (i - i') + (e - e') = i + e - (i' + e')$. Because of the perpendiculars through B and C we have $r = p$, but $p = i' + e' = r$; hence, $d = i + e - r$; that is to say, *the total deviation is equal to the sum of the angles of incidence and emergence diminished by the refracting angle of the prism.*

375. Prism Used for Measuring Refractive Power.—For any given prism the deviation will depend upon the angles of incidence and emergence.

If a prism rotate about an axis parallel to its refracting edge, a position of minimum deviation will be found such that any rotation either to right or left will increase the deviation of the ray; if now the angles of incidence and emergence be measured, they will be found equal.

From the equations

$$r = i' + e'$$

$$d = i + e - r,$$ by making $i = e$, and consequently $i' = e'$, we obtain

$$i = \tfrac{1}{2}(r + d), \text{ and } i' = \tfrac{1}{2} r;$$

from which we find the relative index of refraction

$$m = \frac{\text{sine } i}{\text{sine } i'} = \frac{\text{sine } \frac{1}{2}(r + d)}{\text{sine } \frac{1}{2} r}.$$

Thus having measured the refracting angle of the prism and the minimum deviation of the ray we can at once determine the index of refraction of the substance of which the prism is formed.

If the angle r be very small, d will also be small, and the ratio of the angles may be used instead of the ratio of their sines, and the formula then becomes $m = \frac{r + d}{r} = 1 + \frac{d}{r}$.

This is one of the best methods by which to determine the index of refraction of a solid, transparent substance.

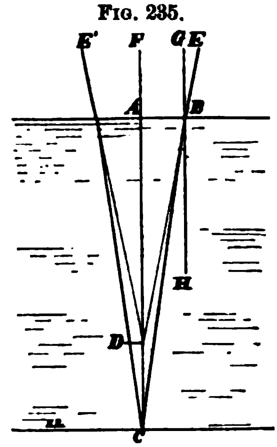
The final deviation of the ray being unaffected by its passage through glass plates with parallel faces, hollow prisms formed of such plates may be filled with a liquid whose index of refraction is to be determined. A tube whose end sections are glass planes equally inclined to the axis of the tube, may be used to determine the relative indices of gases and air, and by exhausting the tube to form a vacuum, the absolute indices may be found.

376. Light through One Surface.—

1. *Plane Surface.* When *parallel* rays pass into another medium through a plane surface, they remain parallel. For the per-

pendiculars being parallel, the angles of incidence are equal, and therefore the angles of refraction are equal also, and the refracted rays parallel. But a pencil of *diverging* rays is made to diverge *less*, when it *enters a denser medium*. For the outer rays make the largest angles of incidence, and are therefore most refracted toward the perpendiculars, and thus toward parallelism with each other. And when *diverging* rays *enter a rarer medium*, they diverge *more;* because the outside rays make the largest angles of incidence, and therefore the largest angles of refraction, by which means they spread more from each other.

FIG. 235.

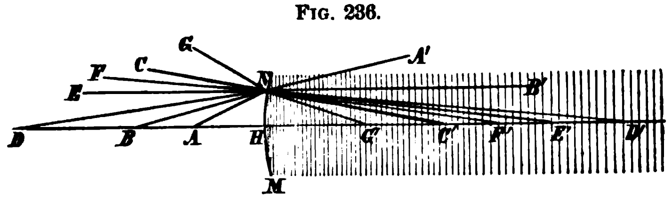


The last case is illustrated when we look perpendicularly into water, and see its depth apparently diminished by about one-fourth of the whole. Let $A\ B$ (Fig. 235) be the surface, and C a point at the bottom, from which pencils come to the eyes at E, E'. They will appear to come from D. As the distance between the pupils of the eyes is less than $2\frac{1}{2}$ inches, the obliquity of the pencil $C\ B\ E$ will be very slight. Let $C\ F$ be perpendicular to the surface $A\ B$.

The angle $C = C\ B\ H =$ angle of incidence; and $A\ D\ B = G\ B\ E =$ angle of refraction. Now, in the triangle $B\ D\ C$, $B\ C : B\ D$ ($:: A\ C : A\ D$ nearly) $::$ sin D : sin C :: sine of refraction : sine of incidence :: 1.34 : 1. Hence the apparent depth is one-fourth less than the real depth. The apparent depth of water may be diminished much more than this by looking into it obliquely.

2. *Convex surface of the denser. A convex surface tends to converge rays.* Let C' (Fig. 236) be the centre of convexity, and

FIG. 236.



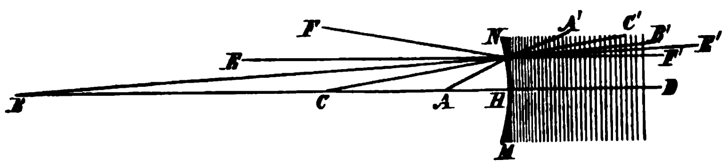
$C'\ D$, $C'\ C$, two radii produced. As rays are bent *toward* the perpendiculars in entering a denser medium, and as the perpendiculars themselves converge to C' the general effect of such a surface is to produce convergency. The pencil, $A\ H$, $A\ N$, is merely made less divergent, $H\ D'\ N\ A'$; $B\ H$, $B\ N$ become parallel,

H D', *N B'*; *D H*, *D N*, convergent to *D'*; the parallel rays, *D H*, *E N*, convergent to *E'*; the convergent pencil, *D H*, *F N*, more convergent to *F'*; but *D H*, *C N*, which converge equally with the radii, are not changed; and *D H*, *G N*, which converge more than the radii, converge less than before, to *G'*. The two last cases, which are exceptions to the general effect, rarely occur in the practical use of lenses.

If we trace in the opposite direction the rays, *A'*, *B'*, *D'*, &c., comparing each with *D' D*, we find, in this case also, that the convex surface tends to converge the rays, by bending them *from* their respective perpendiculars.

3. *Concave surface of the denser. A concave surface tends to diverge rays.* Let *C C'*, *C D* (Fig. 237), be the radii of concavity produced. As the radii *diverge* in the direction in which the light

Fig. 237.

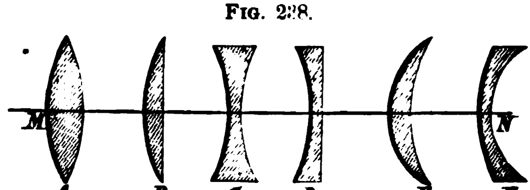


moves, the rays, being bent *toward* them, will generally be made to diverge also. Hence, parallel rays, *B H*, *E N*, are diverged, *H D*, *N E'*; and diverging rays, *B H*, *B N*, are diverged more, *H D*, *N B'*. If, however, rays diverge as much as the radii, or more, they proceed in the same direction, or diverge less, a case which rarely occurs.

If the rays are traced in the opposite direction, the tendency in general to produce divergency appears from the fact that the perpendiculars are now *converging* lines, and the rays are refracted *from* them.

377. Lenses.—A *lens* is a transparent medium bounded by curved surfaces whose centres of curvature lie upon a normal common to the two surfaces. If the radius of curvature is made

Fig. 238.



infinite, the corresponding surface becomes a plane. The usual varieties are shown in Fig. 238.

A *double convex lens* (A) consists of two spherical segments, either equally or unequally convex, having a common base.

A *plano-convex lens* (B) is a lens having one of its sides convex and the other plane, being simply a segment of a sphere.

A *double concave lens* (C) is a solid bounded by two concave spherical surfaces, which may be either equally or unequally concave.

A *plano-concave lens* (D) is a lens one of whose surfaces is plane and the other concave.

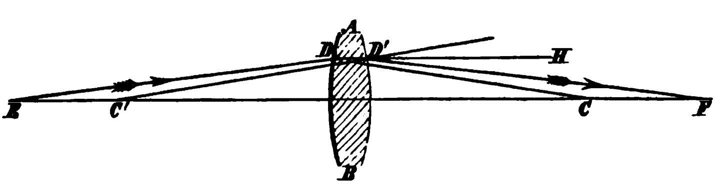
A *meniscus* (E) is a lens one of whose surfaces is convex and the other concave, but the concavity being less than the convexity, it takes the form of a crescent, and has the effect of a convex lens whose convexity is equal to the difference between the sphericities of the two sides.

A *concavo-convex lens* (F) is a lens one of whose surfaces is convex and the other concave, the concavity exceeding the convexity, and the lens being therefore equivalent to a concave lens whose concavity is equal to the difference between the sphericities of the two sides.

A line ($M\ N$) passing through a lens, perpendicular to its opposite surfaces, is called the *axis*. The axis usually, though not necessarily, passes through the centre of the figure.

378. General Effect of the Convex Lens.—Whether double-convex or plano-convex, its general effect is to converge light. It has been shown (Art. 376) that the convex surface of a denser medium tends to converge rays, whichever way they pass through it. Therefore, if E (Fig. 239) is a radiant, while $E\ C'\ C$ follows

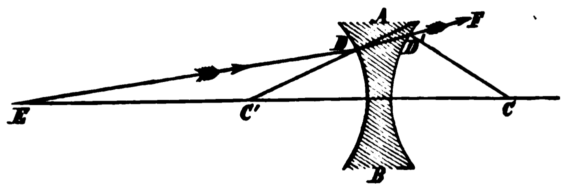
Fig. 239.



the axis without change of direction, the oblique ray $E\ D$ is first refracted *toward* $D\ C$, and then *from* $C'\ D'$ produced, and both actions conspire to converge it to the axis. The rays are represented as meeting in the focus F. Whether the rays are *actually* converged, depends on their previous relation to each other. If the lens is *plano-convex*, the plane surface has usually but little effect in converging the light; but by Art. 376 it may be shown that its action will usually conspire with that of the convex surface.

379. General Effect of the Concave Lens.—This lens, whether double-concave or plano-concave, tends to produce *divergency*. This is evident from what has been shown in Art. 376. The ray $E\,D$ (Fig. 240), in entering the denser medium, is first

FIG. 240.



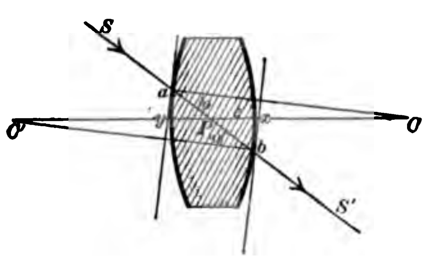
refracted *toward* $C'\,D$ produced, and on leaving the medium at D', is refracted *from* $D'\,C$; and is thus twice refracted from the ray $E\,C$, which being in the axis, is not refracted at all. If the lens is *plano-concave*, the effect of the plane surface may, or may not, conspire with that of the concave surface.

380. The Optic Centre of a Lens.—The incident and emergent portions of a ray which enters and leaves a lens at the points of contact of parallel tangent planes will be parallel according to Art. 372.

The point where the part of such ray included between the bounding surfaces cuts the axis of the lens, or would cut it if produced, is called the optic centre.

In Fig. 241 let a and b be points of contact of parallel tangent planes, then the radii $C\,a$ and $C'\,b$ being perpendicular to these parallel planes are themselves parallel, hence the angles o and o are equal; the angles at P are also equal, and hence the triangles $C\;a\;P$ and $C'\;b\;P$ are similar, and $C'\,P : C\,P :: C'\,b : C\,a$.

FIG. 241.



Represent the thickness of the lens $x\,y$, measured on the axis, by t, and the distance from P, the optic centre, to the surface x by e; also make the radius $C\,a = r$ and $C'\,b = r'$. Substituting these values above we have

$$r' - e : r - (t - e) :: r' : r,$$

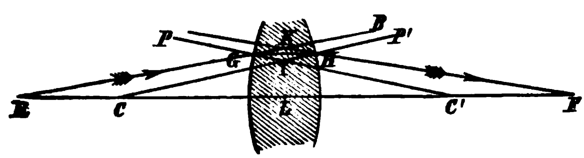
from which we obtain

$$e = \frac{r'\,t}{r' + r} = \frac{r'}{\frac{r'}{t} + \frac{r}{t}}.$$

But this value of e is constant since r, r' and t are constant; therefore all rays which suffer no deviation in passing through the lens must pass through a common point P, called the optic centre. The optic centre is within the lens in the cases of double concave and double convex lenses, but without in the meniscus and concavo-convex. If $r = r'$ the optic centre is midway between the faces.

381. Conjugate Foci.—If the rays from R (Fig. 242) are collected at F, then rays emanating from F will be returned to R; and the two points are called *conjugate foci*. Their relative distances from the lens may be determined when the radii of the

FIG. 242.



surfaces and the index of refraction are known. Let n be the index of refraction, and assume, what is practically true, that the angles of incidence and refraction are so small that their ratio is the same as the ratio of their sines. Then

$$R\,G\,P\,(=K\,G\,I) : I\,G\,H :: n : 1;$$
$$\therefore \quad K\,G\,H : I\,G\,H :: n-1 : 1;$$

in like manner

$$K\,H\,G : I\,H\,G :: n-1 : 1;$$
$$\therefore K\,G\,H + K\,H\,G : I\,G\,H + I\,H\,G :: n-1 : 1.$$

But $K\,G\,H + K\,H\,G = B\,K\,F = R + F$;

and $I\,G\,H + I\,H\,G = G\,I\,C = C + C'$;

naming the acute angles at R, C, C' F, by those letters respectively,

$$\therefore R + F : C + C' :: n - 1 : 1.$$

Now, the lens being thin, and the angles R, C, C', and F very small, the same perpendicular to the axis, at L, the centre of the lens, may be considered as subtending all those angles. Hence, each angle is as the reciprocal of its distance from L. Let $R\,L = p$; $F\,L = q$; $C\,L = r$; and $C'\,L = r'$. Then the equation above becomes,

$$\frac{1}{p} + \frac{1}{q} : \frac{1}{r} + \frac{1}{r'} :: n - 1 : 1;$$

which expresses in general the relation of the conjugate foci.

382. To Find the Principal Focus.—The radiant from which parallel rays come is at an infinite distance. Therefore,

making $p = \infty$, and the distance of the principal focus $= F$, we have $\frac{1}{p} = 0$, and

$$\frac{1}{F} : \frac{1}{r} + \frac{1}{r'} :: n - 1 : 1.$$

If the curvatures are equal, for crown-glass, for which $n = \frac{3}{2}$, F reduces to r; that is, the principal focus of a double convex lens of crown-glass, having equal curvatures, is at the centre of convexity.

The foregoing formulæ are readily adapted to the other forms of lens. When a surface is plane, its radius is infinite, and $\frac{1}{r}$, or $\frac{1}{r'} = 0$. When concave, its centre is thrown upon the same side as the surface, and its radius is to be called negative. And if the focal distance, as given by the formula, becomes negative, it is understood to be on the same side as the radiant; that is, the focus is a virtual radiant.

383. Powers of Lenses Practically Determined.—The reciprocal of the principal focal length of a lens $\frac{1}{F}$, is called the *power of a lens.* From Art. 381 we find

$$\frac{1}{p} + \frac{1}{q} = (n - 1) \left(\frac{1}{r} + \frac{1}{r'}\right),$$

and from Art. 382

$$\frac{1}{F} = (n - 1)\left(\frac{1}{r} + \frac{1}{r'}\right);$$

whence we have

$$\frac{1}{F} = \frac{1}{p} + \frac{1}{q}.$$

As the index of refraction and the radii of curvature are not generally known in respect to any particular lens which we may happen to be using, some practical method by which to determine F will enable us to calculate readily either p or q, the other being given.

(1.) *To find F for a convex lens.*—Form an image of the sun upon a plate of ground glass, and measure the distance of the image from the lens. Or, place a light on one side of the lens and find its sharp image upon a screen on the other side. These distances measured, give p and q, whence $F = \frac{p\,q}{p + q}$.

(2.) These two methods assume the thickness of the lens to be small compared with the focal length. The focal length of a

thick lens, or system of lenses, may be found thus: On one side, at a distance a little greater than F, place a scale strongly illuminated by transmitted light, and receive the sharp and greatly magnified image of one of its divisions upon a screen upon the other side of the lens or lenses. Then let $l =$ length of one division, $L =$ length of its image, $p =$ distance of the screen from the lens (very great compared with its thickness), and we find, from similar right-angled triangles, $L : l :: p : \frac{p\,l}{L} = q$, and these values of p and q give

$$F = \frac{p\,q}{p+q} = \frac{p\,\frac{p\,l}{L}}{p + \frac{p\,l}{L}} = p\,\frac{l}{L+l}.$$

The focal length is strictly the distance from F to the intersection of the axis by the *principal plane* of the lens or combination of lenses.

The *principal plane* passes through the point of intersection of an incident ray, parallel to the axis and its emergent ray, both produced if necessary, and is at right angles to the axis.

(3.) To find F for a concave lens. Use in contact with the concave lens a stronger convex, of known value for F, and proceed according to the preceding methods. Then if $f =$ focal length of combination and $f' =$ focal length of convex alone, and $F =$ that of concave lens sought, we shall find

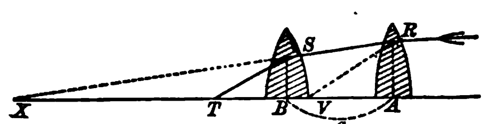
$\frac{1}{F} = \frac{1}{f} - \frac{1}{f'}$, as will be proved hereafter; or when the lens is deep and not very small, take for the focal length that distance from a screen at which the circle of light from the sun is twice the diameter of the lens.

384. Equivalent Combinations.—

To find the focal length of a lens which shall be equal to a combination of two lenses.

Suppose the lenses (Fig. 243) to be of such thickness as may be neglected. Let a ray parallel to the common axis be incident at R. If $R\ V$ be drawn parallel to $S\ T$, the emergent ray, $A\ V$ will represent the focal length, F, of a lens which would produce the same deviation as this combination. Let $A\ X = f =$ focal length of A, then $B\ X = f - a$, a being the distance between B and A.

FIG. 243.



Now if we regard T as a radiant, and TSR as the path of the ray, then X is the virtual conjugate focus of the lens B corresponding to T, and calling f' the focal length of B, we have

$$\text{Art. 383, } \frac{1}{f'} = \frac{1}{BT} - \frac{1}{BX}.$$

Substituting the value of BX above, we have

$$\frac{1}{BT} = \frac{1}{f'} + \frac{1}{f-a} = \frac{f'+f-a}{f'(f-a)}.$$

By similar triangles AVR, BTS and XAR, XBS

$$\frac{BT}{AV(=F)} = \frac{BS}{AR} = \frac{BX}{AX}, \text{ whence}$$

$$\frac{1}{F} = \frac{f-a}{f} \times \frac{f'+f-a}{f'(f-a)} = \frac{f'+f-a}{ff'}$$

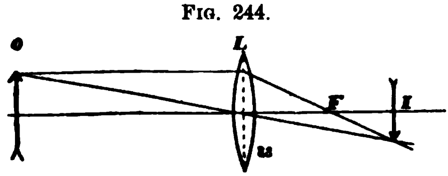
$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} - \frac{a}{ff'}.$$

When the lenses are in contact the distance $a = o$, and we have $\frac{1}{F} = \frac{1}{f} + \frac{1}{f'}$; that is to say,

The power of a combination of two lenses in contact is equal to the sum of their respective powers.

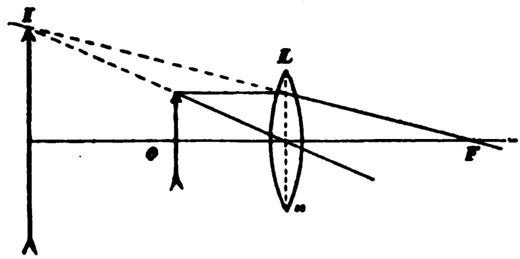
385. Images by the Convex Lens.—The *convex* lens forms a variety of images, whose character and position depend on the place of the object. If the position of the object and the focal length of the lens are known the position of the image can be readily determined. Light reflected from any point of the object will be converged to the conjugate focus of that point. By determining one such conjugate focus we can determine the position and size of the image. This one may be found by merely tracing the path of two rays from the same point of the object—one ray parallel to the axis and the other through the optic centre of the lens. The first will be refracted to the principal focus and the latter will not be deviated. The point of intersection of these rays is the required conjugate focus. In the next three diagrams let L be the lens, F the principal focus, O the object, and I the image. In Fig. 244 let the object be at a greater distance from the lens than the focal length. The two rays from the head of the object intersect at the head of the image. Being below the axis shows that the image is inverted, and being nearer to the lens than object shows that it is smaller. In Fig. 245 the

FIG. 244.



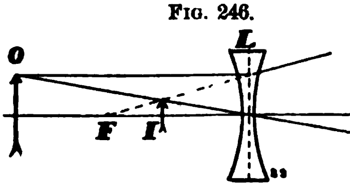
object is placed between the principal focus and the lens. That the two rays may intersect, they must be produced beyond the lens. The image is then not real but *apparent*, *i.e.*, rays from one point of the object do not come to a conjugate focus, but appear to come from the corresponding point of the image.

FIG. 245.



386. Images by the Concave Lens.—The images by concave lenses may be studied in the same manner. In drawing the ray which is parallel to the axis, however, instead of converging it to the focus on the opposite side of the lens, it must be *diverged from* the focus which is on the same side as the object. The process is shown in Fig. 246. Images from concave lenses are always apparent, erect and smaller than the objects.

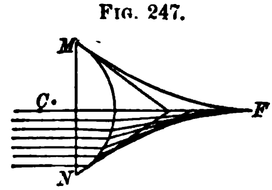
FIG. 246.



It is noticeable that the *concave mirror* and the *convex lens* are analogous in their effects, forming images on both sides, both real and apparent, both erect and inverted, both larger and smaller than the object; while the *convex mirror* and the *concave lens* also resemble each other, producing images always on one side, always apparent, always erect, always smaller than the object.

387. Caustics by Refraction.—If the convex surface of a lens is a considerable part of a hemisphere, the rays more distant from the axis will be so much more refracted than others, as to cross them and meet the axis at nearer points, thus forming caustics by refraction. Fig. 247 shows this effect in the case of parallel rays; those near the axis intersecting it at the principal focus *F*, and the intersections of remoter rays being nearer and nearer to the lens, so that the whole converging pencil assumes a form resembling a cone with concave sides.

FIG. 247.



The grating (Fig. 248), viewed through such a lens, would

appear distorted, as in Fig. 249, and if viewed through a concave lens the opposite effect would result, as in Fig. 250.

FIG. 248.

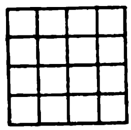


FIG. 249.

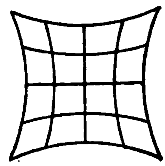
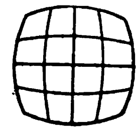
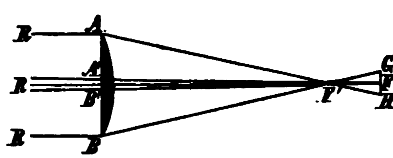


FIG. 250.



388. Spherical Aberration of a Lens.—The production of caustics is an extreme case of what is called spherical aberration. Unless the lens is of small angular breadth, not more than 10°, a pencil whose rays originated in one point of an object is not converged accurately to one point of the image, but the outer rays are refracted too much, and make their focus nearer the lens than that of the central rays, as represented in Fig. 251. If F is the focus of the central rays, and F' of the extreme ones, other rays of the same beam are collected in intermediate points, and $F\,F'$ is called the *longitudinal spherical aberration;* and $G\,H$, the breadth covered by the pencil at the focus of central rays, is called the *lateral spherical aberration.*

FIG. 251.



Such a lens cannot form a distinct image of any object; because perfect distinctness requires that all rays from any one point of the object should be collected to one point in the image. If, for example, the beam whose outside rays are $R\,A$, $R\,B$, comes from a point of the moon's disc, that point will not be perfectly represented by F, because a part of its light covers the circle, whose diameter is $G\,H$, thus overlapping the space representing adjacent points of the moon. And if that point had been on the edge of the moon's disc, F could not be a point of a well-defined edge of the image, since a part of the light would be spread over the distance $F\,G$ outside of it, and destroy the distinctness of its outline.

389. Remedy for Spherical Aberration.—As spherical lenses refract too much those rays which pass through the outer parts, it is obvious that, to destroy aberration, a lens is required whose curvature diminishes toward the edges. Accordingly, forms for *ellipsoidal* lenses have been calculated, which in theory will completely remove this species of aberration. But no curved

solids can be so accurately ground as those whose curvature is uniform in all planes, that is, the spherical. Hence, in practice it is found better to *reduce* the aberration as much as possible by spherical lenses, than to attempt an entire *removal* of it by other forms which cannot be well made.

Lenses, or combinations which are free from spherical aberration, are said to be *Aplanatic*. By lessening the aperture of a lens by a suitable diaphragm, the aberration may be much diminished.

In a plano-convex lens, whose plane surface is toward the object, the spherical aberration is 4.5; that is (Fig. 251), $F\,F' = 4.5$ times the thickness of the lens. But the same lens, with its convex side toward the object, is far better, its aberration being only 1.17. In a double convex lens of equal curvatures, the aberration is 1.67; if the radii of curvature are as 1 : 6, and the most convex side is toward the object, the aberration is only 1.07. By placing two plano-convex lenses near each other, the aberration may be still more reduced.

390. Atmospheric Refraction.—The atmosphere may be regarded as a transparent spherical shell, whose density increases from its upper surface to the earth. The radii of the earth produced are the perpendiculars of all the laminæ of the air; and rays of light coming from the vacuum beyond, if oblique, are bent *gradually* toward these perpendiculars; and therefore heavenly bodies appear more elevated than they really are. The greatest elevation by refraction takes place at the horizon, where it is about half a degree.

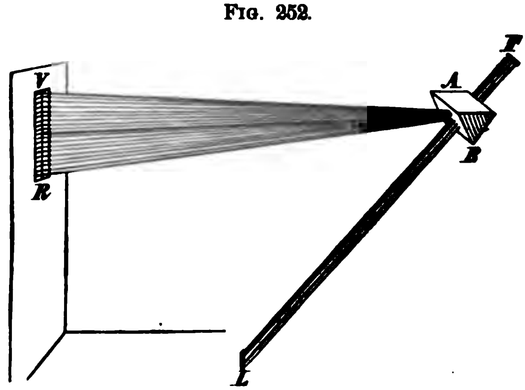
391. Mirage.—This phenomenon, called also *looming*, consists of the formation of one or more images of a distant object, caused by horizontal strata of air of very different densities. Ships at sea are sometimes seen when beyond the horizon, and their images occasionally assume distorted forms, contracted or elongated in a vertical direction. These effects are generally ascribed to *extraordinary refraction* in horizontal strata, whose difference of density is unusually great. But many cases of mirage seem to be instances of *total reflection* from a highly rarefied stratum resting on the earth. These occur frequently on extended sandy plains, as those of Egypt. When the surface becomes heated, distant villages, on more elevated ground, are seen accompanied by their images inverted below them, as in water. As the traveler advances, what appeared to be an expanse of water retires before him. By placing alcohol upon water in a glass vessel, and allowing them time to mingle a little at their common surface, the phenomena of mirage may be artificially represented.

CHAPTER IV.

DECOMPOSITION AND DISPERSION OF LIGHT.

392. The Prismatic Spectrum.—Another change which light suffers in passing into a new medium, is called *decomposition*, or the separation of light into colors. For this purpose, the glass prism is generally employed. It is so mounted on a jointed stand, that it can be placed in any desired position across the beam from the heliostat. The beam, as already noticed, is bent away from the refracting angle, both in entering and leaving the prism, and deviates several degrees from its former direction. If the light is admitted through a narrow aperture, *F* (Fig. 252), and

FIG. 252.



the axis of the prism is placed parallel to the length of the aperture, the light no longer falls, as before, in a narrow line, *L*, but is extended into a band of colors, *R V*, whose length is in a plane at right angles to the axis of the prism. This is called the prismatic spectrum. Its colors are usually regarded as seven in number—*red, orange, yellow, green, blue, indigo, violet*. The red is invariably nearest to the original direction of the beam, and the violet the most remote; and it is because the elements of white light are unequally refrangible, that they become separated, by transmission through a refracting body. The spectrum is properly regarded as consisting of innumerable shades of color. Instead of Newton's division into *seven* colors, many choose to consider all the varieties of tint as caused by the combination of *three* primitive colors, *red, yellow,* and *blue,* varying in their pro-

portions throughout the entire spectrum. The number *seven*, as perhaps any other particular number, must be regarded as arbitrary.

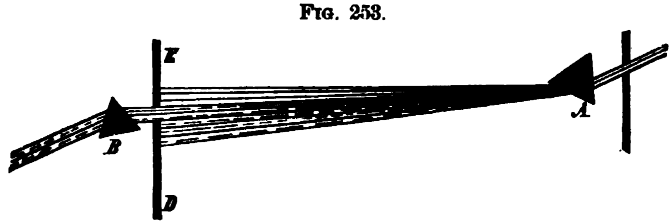
The spectrum contains rays of other wave lengths than those which affect the eye. The rays of longest wave length are crowded together at and beyond the red, and here the greatest heat is found upon testing with a thermometer.

The chemical or actinic rays of shortest wave lengths, are found at and beyond the violet. These invisible rays differ from those which are visible only in wave length.

Light from other sources is also susceptible of decomposition by the prism; but the spectrum, though resembling that of the sun, usually differs in the proportion of the colors.

393. The Individual Colors of the Spectrum cannot be Decomposed by Refraction.—If the spectrum formed by the prism A be allowed to fall on the screen $E\ D$ (Fig. 253), and one color of it, green for example, be let through the screen, and

FIG. 253.



received on a second prism, B, it is still refracted as before, but all its rays remain together and of the same color. The same is true of every color of the spectrum. Therefore, so far as refrangibility is concerned, all the colors of the spectrum are alike simple.

394. Colors of the Spectrum Recombined.—It may be shown, in several ways, that if all the colors of the spectrum be combined, they will *reproduce white light*. One method is by transmitting the beam successively through two prisms whose refracting angles are on opposite sides. By the first prism, the colors are separated at a certain angle of deviation, and then fall on the second, which tends to produce the same deviation in the opposite direction, by which means all the colors are brought upon the same ground, and the illuminated spot is white as if no prism had been interposed. Or the colors may be received on a series of small plane mirrors, which admit of such adjustment as to reflect all the beams upon one spot. Or finally, the several colors can, by different methods, be passed so rapidly before the eye that their

visual impressions shall be united in one; in which case the illuminated surface appears white.

395. Complementary Colors.—If certain colors of the spectrum are combined in a compound color, and the others in another, these two are called *complementary colors*, because, when united, they will produce white. For example, if *green*, *blue*, and *yellow* are combined, they will produce green, differing slightly from that of the spectrum; the remaining colors, *red*, *orange*, *indigo*, and *violet*, compose a kind of purple, unlike any color of the spectrum. But these particular shades of *green* and *purple*, if mingled, will make perfectly white light, and are therefore complementary colors.

Tyndal gives these as complementary: Red and greenish blue, orange and cyanogen blue, yellow and indigo blue, greenish yellow and violet.

396. Natural Colors of Bodies.—The colors which bodies exhibit, when seen in ordinary white light, are owing to the fact that they decompose light by absorbing or transmitting some colors and reflecting the others. We say that a body *has* a certain color, whereas it only *reflects* that color; a flower is called red, because it reflects only or principally red light; another yellow, because it reflects yellow light, &c. A white surface is one which reflects all colors in their due proportion; and such a surface, placed in the spectrum, assumes each color perfectly, since it is capable of reflecting all. A substance which reflects no light, or but very little, is black. What peculiarity of constitution that is which causes a substance to reflect a certain color, and to absorb others, is unknown.

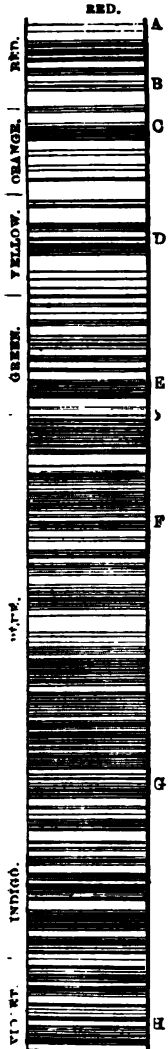
Very few objects have a color which exactly corresponds to any color of the spectrum. This is found to result from the fact that most bodies, while they reflect some one color chiefly, reflect the others in some degree. A red flower reflects the red light abundantly, and perhaps some rays of all the other colors with the red. Hence there may be as many shades of red as there can be different proportions of other colors intermingled with it. The same is true of each color of the spectrum. Thus there is an infinite variety of tints in natural objects. These facts are readily established by using the prism to decompose the light which bodies reflect.

397. The Continuous Spectrum.—A spectrum which contains all the different shades of color is called a *continuous spectrum*. It may be obtained from the light emitted by incandescent solids or liquids. The molecules of elementary solids have their

own rate of vibration, and, if allowed to vibrate in an unobstructed path, would communicate this rate to the ether. They, however, do not have an unobstructed path, but are continually colliding with each other, and, as a consequence, communicate to the ether vibrations of varying frequencies. In a short time they furnish all the frequencies which are represented by the different parts of the spectrum. Owing to persistence of vision, the eye is incapable of detecting the absence of certain frequencies at a given instant.

Fig. 254.

RED.



VIOLET.

398. The Line Spectrum.—If the light, emitted by the incandescent vapor of a chemical element, be passed through a narrow aperture and a prism, it will give a *line spectrum*, *i.e.*, instead of all the colors a few narrow bright lines will be seen. The lines are parallel to the axis of the prism. Every element has its own set of lines. A sodium salt placed at the edge of the colorless flame of a Bunsen burner will give the flame a yellow shade, and its spectrum will consist of two narrow, yellow lines. A lithium salt gives a red line. The paths of the molecules, when in the form of vapor, are unobstructed. Only definite rates of vibration are communicated to the ether, giving definite colors. The spectrum, then, must be a line spectrum. When an element gives two or more lines, it is reasonable to suppose that its molecules vibrate harmonically.

399. Frauenhofer Lines.—If white light be passed through a vapor of a chemical element, which, however, is not incandescent, those vibrations which correspond to the frequencies of the vapor molecules, will be absorbed by the vapor. An examination of the spectrum will reveal that it is no longer continuous, but has dark lines in the color which corresponds to the frequencies of the vibrations of which it has been robbed. This phenomena has its parallel in sound. A screen of wires, tuned to the same pitch as an organ pipe and interposed between the pipe and a listener, will absorb the vibrations of the pipe so as to make them inaudible.

The spectrum of the sun contains a great number of these dark lines, and they are called *Frauenhofer lines* after the name of their discoverer. Their appearance is roughly represented in Fig. 254.

The positions of over six thousand have been determined. The non-luminous vapors which surround the sun absorb the vibrations corresponding to these lines from the mixture of vibrations sent out by the incandescent centre.

From what has been said it will be seen that the spectrum furnishes us with a means of determining the constitution of the vapor envelope of the sun or of stars. We have to map out line spectra of all the chemical elements, and then match them into the dark lines in the solar spectrum. Thus luminous sodium vapor gives two lines in yellow; solar spectrum has two equally broad dark lines at the same place in the yellow; hence the sun's light passes through sodium vapor.

The spectra of the same substance at widely different temperatures are often remarkably dissimilar. At the very high temperature near the solid or liquid nucleus of the sun, the chemical elements themselves appear to be broken up and reduced to simpler forms of matter. This, together with the fact that there are many coincidences between the lines of the spectra of different elements, has given rise to a theory that all the different chemical elements are but modifications of a single kind of matter.

400. The Spectroscope.—This instrument is used in examining and comparing spectra. It consists of a round horizontal stand upon which is placed one or more prisms. Projecting from the table are three telescopes. One of these contains an adjustable aperture to receive the light to be examined. The light traversing it is refracted by the prism and enters a second telescope, at the end of which is placed the observer's eye. The third telescope contains a transparent scale. Light from an auxiliary source, *e.g.*, a candle, passes through the scale and is reflected from one surface of the prism into the observing telescope. The observer thus sees a spectrum and scale overlapping or adjoining it.

401. Dispersion of Light.—Decomposition of light refers to the *fact* of a separation of colors; *dispersion*, rather, to the *measure* or *degree* of that separation. The *dispersive power* of a medium indicates the amount of separation which it produces, compared with the amount of refraction.

The deviation of the line E is usually taken as the deviation of the beam regarded as a whole. The differenee of the deviations of the lines A and H is the dispersion.

For example, if a substance, in refracting a beam of light $1^\circ\ 51'$ from its course, separates the violet from the red by $4'$, then its dispersive power is $\frac{4}{111} = .036$. The following table

gives the dispersive power of a few substances much used in optics:

	Dispersive power.		Dispersive power.
Oil of cassia..............	0.139	Plate-glass................	0.032
Sulphuret of carbon.......	0.130	Sulphuric acid............	0.031
Oil of bitter almonds.......	0.079	Alcohol...................	0.029
Flint-glass................	0.052	Rock-crystal..............	0.026
Muriatic acid.............	0.043	Blue sapphire	0.026
Diamond..................	0.038	Fluor-spar................	0.022
Crown-glass..............	0.036		

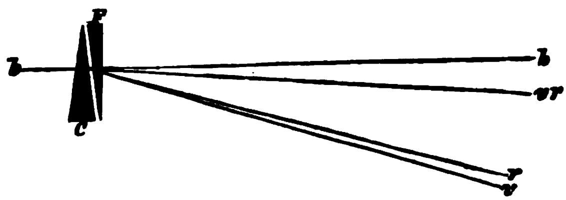
The discovery that different substances produce different degrees of dispersion, is due to Dollond, who soon applied it to the removal of a serious difficulty in the construction of optical instruments.

402. Chromatic Aberration of Lenses.—This is a deviation of light from a focal point, occasioned by the different refrangibility of the colors. If the surface of a lens be covered, except a narrow ring near the edge, and a sunbeam be transmitted through the ring, the chromatic aberration becomes very apparent; for the most refrangible color, violet, comes to its focus nearest, and then the other colors in order, the focus of red being most remote. Since the distinctness of an image depends on the accurate meeting of rays of the same pencil in one point, it is clear that discoloration and indistinctness are caused by the separation of colors.

403. Achromatism.—In order to refract light, and still keep the colors united, it is necessary that, after the beam has been refracted, and thus separated, a substance of greater dispersive power should be used, which may bring the colors together again, by refracting the beam only a part of the distance back to its original direction. For instance, suppose two prisms, one of crown-glass and one of flint-glass, each ground to such a refracting angle as to separate the violet from the red ray by 4′. In order for this, the crown-glass, whose dispersive power is .036, must refract the beam 1° 51′; for $\frac{4'}{1° 51'} = .036$; and the flint-glass, whose dispersive power is .052, must refract only 1° 17′; for $\frac{4'}{1° 17'} = .052$. Place these two prisms together, base to edge, as in Fig. 255, *C* being the crown-glass and *F* the flint-glass. Then *C* will refract the beam *b b*, downward 1° 51′, and the violet, *v*, 4′ more than the red, *r*; *F* will refract this decomposed beam

upward 1° 17′, and the violet 4′ more than the red, which will just bring them together at *v r*. Thus the colors are united again, and

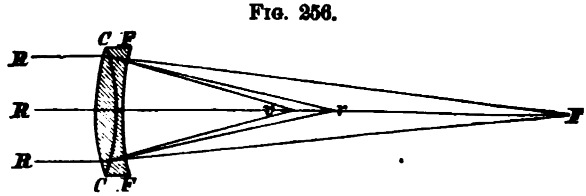
FIG. 255.



yet the beam is refracted downward 1° 51′ — 1° 17′ = 34′, from its original direction.

404. Achromatic Lens.—If two prisms can thus produce achromatism, the same may be effected by lenses; for a convex lens of crown-glass may converge the rays of a pencil, and then a concave lens of flint-glass may diminish that convergency sufficiently to unite the colors. A lens thus constructed of two lenses of different materials and opposite curvatures, so adapted as to produce an image free from chromatic aberration, is called an *achromatic lens.* Fig. 256 shows such a combination. The con-

FIG. 256.



vex lens of crown-glass alone would gather the rays into a series of colored foci from *v* to *r*; the concave flint-glass lens refracts them partly back again, and collects all the colors at one point, *F.*

405. Colors not Dispersed Proportionally.—It is assumed in the foregoing discussion, that when the red and violet are united, all the intermediate colors will be united also. It is found that this is not strictly true, but that different substances separate two given colors of the spectrum by intervals which have different ratios to the whole length of the spectrum. This departure from a constant ratio in the distances of the several colors, as dispersed by different media, is called the *irrationality of dispersion.* In consequence of it there will exist some slight discoloration in the image, after uniting the extreme colors. It is found better in practice to fit the curvatures of the lenses, for uniting those rays which most powerfully affect the eye.

In a well-corrected telescope, when pointed at a bright object, such as Jupiter or the moon, a purple color will be seen when the eye-piece is pushed inwards from its position of adjustment, and a greenish color will show when the eye-piece is pulled out too far.

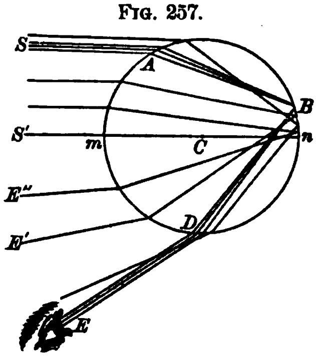
CHAPTER V.

RAINBOW AND HALO.

406. The Rainbow.—This phenomenon, when exhibited most perfectly, consists of two colored circular arches, projected on falling rain, on which the sun is shining from the opposite part of the heavens. They are called the *inner* or *primary* bow, and the *outer* or *secondary* bow. Each contains all the colors of the spectrum, arranged in contrary order; in the primary, red is outermost; in the secondary, violet is outermost. The primary bow is narrower and brighter than the secondary, and when of unusual brightness, is accompanied by *supernumerary* bows, as they are called; that is, narrow red arches just within it, or overlapping the violet; sometimes three or four supernumeraries can be traced for a short distance. The common centre of the bows is in a line drawn from the sun through the eye of the spectator.

407. Action of a Transparent Sphere on Light.—Let a hollow sphere of glass be filled with water, and cause a beam of parallel rays of homogeneous yellow light to fall upon it. To prevent confusion in Fig. 257, we will consider only those rays which fall upon the upper half of the section of the sphere, and will trace them as they emerge at the lower half. Those rays which enter near the axis $S'\ m$ will be refracted to points near n. Rays still farther from $S'\ m$ will be refracted to points still farther from n. Rays at about 59° from m, at A, will be refracted to B, and no ray, no matter where it may enter the sphere, can be refracted to a point higher than B. Now as B is the limit of the arc $n\ B$, it follows that rays close to the middle *ray* of the pencil

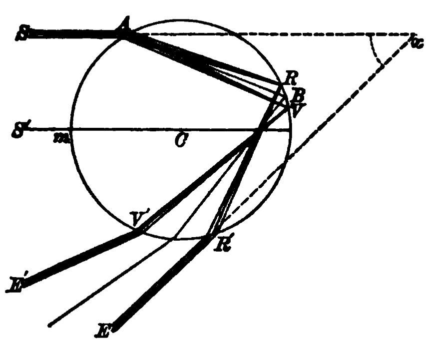
FIG. 257.



$S\,A$, both above it and below it, will be refracted to B, crowded together as it were, and after reflection a large portion will emerge at D. As on passing into the sphere at A from air, these converged to B, so on emerging the reverse action takes place at D, and we have a compact pencil of parallel rays. An eye at E would receive an impression of bright light in the direction $E\,D$; an eye below $E\,D$ would receive no light at all, and at E' or E'', while some light would be received from the diverging rays, the impression would be much less vivid than at E. We have been considering only a section of the sphere through the axis; if now we conceive this section to revolve about the axis $S'\,C$, our beam $S\,A$ becomes a hollow cylinder of light, and the emergent beam $E\,D$ becomes an emergent hollow frustum of a cone, and if the eye be placed at any element of this cone the effect will be as described for the element E.

408. The Primary Bow.—In the preceding article, homogeneous yellow light was considered. Let us examine the results when white light from the sun falls upon a rain-drop. Suppose $S\,A$ (Fig. 258) to be a beam of parallel rays from the sun incident at 59° from m. As B was the point at which yellow rays of the beam were concentrated, the red rays which are less refrangible will all concentrate at R, the distance $R\,B$ being very greatly exaggerated for the sake of clearness in the diagram. After reflection these red rays will emerge as a beam of *parallel red* rays at R'. The violet rays of the beam $S\,A$ being most refrangible will all meet at V, below B, and will emerge at V' as a beam of parallel violet rays. Between these will be beams of the intermediate colors of the spectrum. These are beams of parallel *rays*, but are not parallel *beams*, as is shown in the figure. The angle x included between the incident beam $S\,A$ and the emergent red beam produced backward to x is found by calculation to be 42° 2′, and the like angle for the violet beam is 40° 17′.

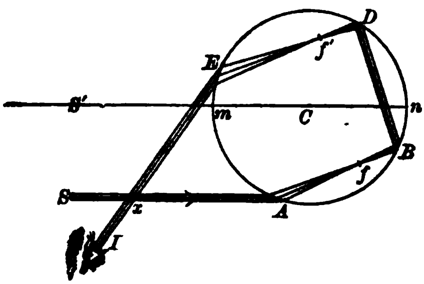
FIG. 258.



409. Course of Rays in Secondary Bow.—If we examine the conditions of two internal reflections (Fig. 259), we find that a beam of monochromatic light entering at a certain

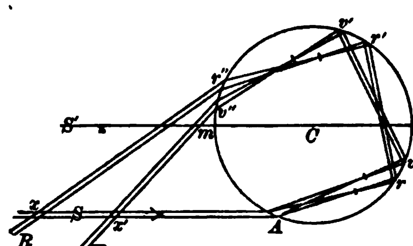
distance from the axis $S' n$, about 71° 42′, suffers the least deviation possible after two reflections, that is to say, the angle $S x I$ will be a minimum. Rays near this ray $S A$ of minimum deviation both on the side towards the axis $S' n$ and on the side away from it will tend to meet in a focus at f about $\frac{3}{4}$ of the distance $A B$, and will then be reflected parallel to D, again being reflected to a focus at f' ($E f' = \frac{3}{4}$ $E D$), and finally emerging at E, a parallel beam as on entering. An observer at I would receive an intense beam of light of the particular color used.

FIG. 259.



Now substitute for the monochromatic light, light from the sun, and the results will be as illustrated in Fig. 260, in which the difference in direction between the red and the violet rays has been greatly exaggerated. At A the red rays, following the course given in Fig. 260 are converged, cross at the focus, and at r are reflected as a parallel beam to r'; here they are again reflected to a focus, and again diverging pass on to r'', where they emerge as a parallel beam $r'' R$. The violet rays are separated from the red at A, and being more refrangible, take the path indicated in the figure. The angle $A x r'' = 50° 59'$ and $A x' v'' = 54° 9'$. In order that the emergent pencil may enter the eye the incident ray must enter on the side of the drop nearest the observer. The rays just considered are the only ones which, after *two* reflections, emerge compact and parallel, and give bright color at a great distance.

FIG. 260.



The explanation just concluded gives a general and sufficiently exact conception of the phenomena of the primary and secondary bows.

A rigid mathematical analysis would take note of the caustics by reflection and refraction, and would vary slightly the places of maximum illumination. To such analysis, and to the principle of interference, the student must refer for an account of the super-

numerary bows accompanying the primary, which are sometimes observed.

410. Axis of the Bows.—Let *A B D G I* (Fig. 261) represent the path of the pencil of red light in the primary bow. If *A B* and *I G* are produced to meet in *K*, the angle *K* is the deviation, 42° 2′, of the incident and emergent red rays. Suppose the spectator at *I*, and let a line from the sun be drawn through his position to *T*; it is sensibly parallel to *A B*, and therefore the angles *I* and *K* are equal. As *T* is opposite to the sun, the red

FIG. 261.

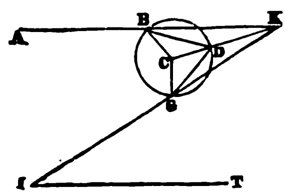
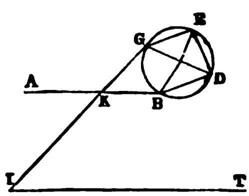


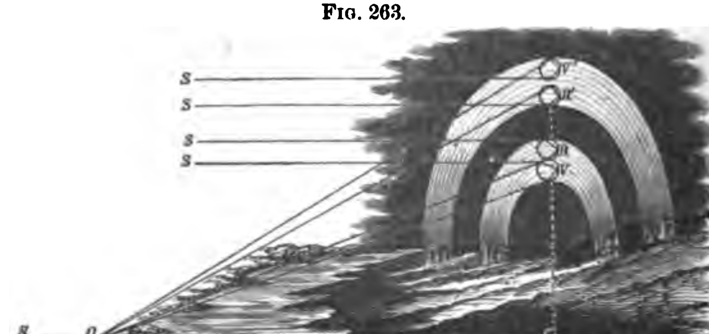
FIG. 262.



color is seen at the distance of 42° 2′, on the sky, from the point *T*; and so the angular distance of each color from *T* equals the angle which the ray of that color makes with the incident ray. In like manner, in the secondary bow, if *I T* (Fig. 262) be drawn through the sun and the eye of the observer, it is parallel to *A B*, and the angular distance of the colored ray from *T* is equal to *K*, the deviation of the incident and emergent rays. *I T* is called the *axis* of the bows, for a reason which is explained in the next article.

411. Circular Form of the Bows.—Let *S O C* (Fig. 263) be a straight line passing from the sun, through the observer's

FIG. 263.



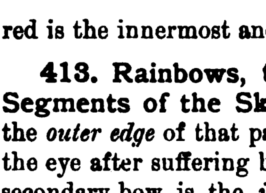
place at *O*, to the opposite point of the sky; and let *V O*, *R O* be the extreme rays, which after *one* reflection bring colors to the eye

at O, and $R'\ O$, $V'\ O$, those which exhibit colors after *two* reflections; then (according to Arts. 408, 409), $V\ O\ C = 40° \ 17'$, $R\ O\ C = 42° \ 2'$, $R'\ O\ C = 50° \ 59'$, $V'\ O\ C = 54° \ 9'$. Now, if we suppose the whole system of lines, $S\ V'\ O$, $S\ V\ O$, to revolve about $S\ O\ C$, as an axis, the relations of the rays to the drops, and to each other, will not be at all changed; and the same colors will describe the same lines, whatever positions those lines may occupy in the revolution. The emergent rays, therefore, all describe the surfaces of cones, whose common vertex is in the eye at O; and the colors, as seen on the cloud, are the circumferences of their bases.

In a given position of the observer, the extent of the arches depends on the elevation of the sun. When on the horizon, the bows are semicircles; but less as the sun is higher, because their centre is depressed as much below the horizon as the sun is elevated above it. From the top of a mountain, the bows have been seen as almost entire circles.

412. Colors of the Two Bows in Reversed Order.—Suppose the eye to receive a red ray from a drop a (Fig. 264); rays of all other colors being more refrangible than the red would pass above the eye, as does v'. In order that a violet ray may enter the eye it must proceed from a lower drop, as b, and the less refrangible rays from this drop will pass below the eye, as at R'. Hence in the primary bow the drops which send violet to the eye are nearer to the axis of the bow than those which send red, red being therefore the outermost color. A like examination of the secondary bow shows that red is the innermost and violet the outermost color.

FIG. 264.



413. Rainbows, the Colored Borders of Illuminated Segments of the Sky.—The primary bow is to be regarded as the *outer edge* of that part of the sky from which rays can come to the eye after suffering but *one* reflection in drops of rain; and the secondary bow is the *inner edge* of that part from which light, after being *twice* reflected, can reach the eye.

It is found by calculation, that in case of one reflection, the incident and emergent rays can make no inclinations with each other greater than 42° 2′ for red light, and 40° 17′ for violet; but the inclinations may be less in any degree down to 0°. Therefore, all light, once reflected, comes to the eye from *within* the primary bow.

But the angles, 50° 59′ and 54° 9′, are, by calculation, the *least*

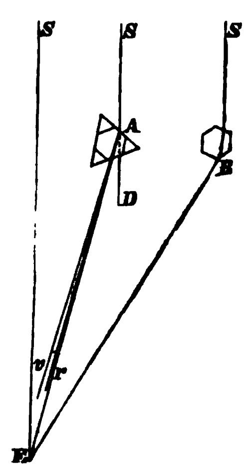
deviations of red and violet light from the incident rays after *two* reflections. But the deviations may be greater than these limits up to 180°. Therefore rays twice reflected can come to the eye from any part of the sky, except between the secondary bow and its centre.

It appears, then, that from the zone lying between the two bows, no light, reflected by drops internally, either once or twice, can possibly reach the eye. Observation confirms these statements; when the bows are bright, the rain within the primary is more luminous than elsewhere; and outside of the secondary bow, there is more illumination than between the two bows, where the cloud is perceptibly darkest.

414. The Common Halo.—This, as usually seen, is a white or colored circle of about 22° radius, formed around the sun or moon. It might, without impropriety, be termed the *frost-bow,* since it is known to be formed by light refracted by crystals of ice suspended in the air. It is formed when the sun or moon shines through an atmosphere somewhat hazy. About the sun it is a white ring, with its inner edge red, and somewhat sharply defined, while its outer edge is colorless, and gradually shades off into the light of the sky. Around the moon it differs only in showing little or no color on the inner edge.

415. How Caused.—The phenomenon is produced by light passing through crystals of ice, having sides inclined to each other at an angle of 60°. Let the eye be at E (Fig. 265), and the sun in the direction $E\ S$. Let $S\ A$, $S\ B$, &c., be rays striking upon such crystals as may happen to lie in a position to refract the light toward $S\ E$ as an axis. Each crystal turns the ray from the refracting edge on entering; and again, on leaving, it is bent still more, and the emergent pencil is decomposed. The color, which comes from each one to the eye E, depends on its angular distance from $E\ S$, and the position of its refracting angle. The angle of deviation for A is $E\ A\ D = S\ E\ A$; for B, it is $S\ E\ B$, and so on. It is found by calculation, that the least deviation for red light is 21° 45′; the least for orange must be a little greater, because it is a little more refrangible, and so on for the colors in order. The greatest deviation for the rays generally is about 43° 13′. All light, therefore, which can be transmitted

FIG. 265.



by such crystals must come to the observer from points somewhere between these two limits, 21° 45′ and 43° 13′ from the sun. But by far the greater part of it, as ascertained by calculation, passes through near the least limit.

416. Its Circular Form.—What takes place on one side of *E S* may occur on every side; or, in other words, we may suppose the figure revolved about *E S* as an axis, and then the transmitted light will appear in a ring about the sun *S*. The inner edge of the ring is red, since that color deviates least; just outside of the red the orange mingles with it; beyond that are the red, orange, and yellow combined; and so on, till, at the minimum angle for violet, all the colors will exist (though not in equal proportions), and the violet will be scarcely distinguishable from white. Beyond this narrow colored band the halo is white, growing more and more faint, so that its outer limit is not discernible at all.

417. The Halo, a Bright Border of an Illuminated Zone.—As in the rainbow, so in the halo, the visible band of colors is only the border of a large illuminated space on the sky. The ordinary halo, therefore, is the bright inner border of a zone, which is more than 20° wide. The whole zone, except the inner edge, is too faint to be generally noticed, though it is perceptibly more luminous than the space between the halo and the luminary.

418. Frequency of the Halo.—The halo is less brilliant and beautiful, but far more frequent, than the rainbow. Scarcely a week passes during the whole year in which the phenomenon does not occur. In summer the crystals are three or four miles high, above the limit of perpetual frost. As the rainbow is sometimes seen in dew-drops on the ground, so the frost-bow, just after sunrise, has been noticed in the crystals which fringe the grass.

419. The Mock Sun.—The mock sun, or sun-dog, is a short arc of the halo, occasionally seen at 22° distance, on the right and left of the sun, when near the horizon. The crystals, which are concerned in producing the mock sun, are supposed to have the form of *spiculæ*, or six-sided *needles*, whose alternate sides are inclined to each other at an angle of 60°; these falling through the air with their axes vertical, refract the light only in directions nearly horizontal, and therefore present only the right and left sides of the halo.

In high latitudes, other and complex forms of halo are frequent, depending for their formation on the prevalence of crystals of other angles than 60°. [See Appendix for calculations of the angular radius of rainbows and halo.]

CHAPTER VI.

NATURE OF LIGHT.—WAVE THEORY.

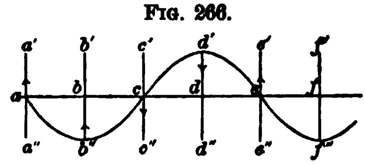
420. The Wave Theory.—Light has sometimes been regarded as consisting of ***material particles*** emanating from luminous bodies. But this, called the ***corpuscular*** or ***emission theory,*** has mostly yielded to the ***undulatory*** or ***wave theory,*** which supposes light to consist of vibrations in a medium. This medium, called the *luminiferous ether,* is imagined to exist throughout all space, and to be of such rarity as to pervade all other matter. It is supposed also to be elastic in a very high degree, so that undulations excited in it are transmitted with great velocity.

There is no independent evidence of the existence of this theoretical ether.

Radiant heat consists of undulations of the same ether, which differ from those of light only in being slower. For it is a familiar fact, that when the heat of a body is increased to about 500° or 600° C. the body becomes luminous, and the brightness increases as the temperature is raised.

421. Nature of the Wave.—Suppose a number of ether molecules, as *a*, *b*, *c*, &c. (Fig. 266), to be equidistant upon the straight line *a f*. Now conceive that *a* moves to *a'*, then back to *a''*, thence to *a* again, occupying four equal intervals of time in the circuit. Suppose *b* to start on the same round, at a time one interval later; when *a* reaches its original position, and is just beginning its upward motion, as in the figure, *b* will be at *b''* moving towards *b*. In like manner, starting *c* and *d* at intervals later by one, we shall find their positions and directions of motion, when *a* begins its second circuit, to be as given in the figure. The motion will have been transmitted to *e*, which will begin its first circuit at the instant that *a* starts upon its second. Molecules which like *a* and *e* are in the same condition as to *place* and *direction of motion,* are said to be in the same phase. A wave length is the distance between two *consecutive* like phases. The amplitude of vibration is the distance between the two limits of the excursion of the particle. It is evident from the figure that the motion is communicated

FIG. 266.



along the axis $a\ f$ a distance $a\ e$, or one wave length, during the time of one vibration.

422. Postulates of the Wave Theory.—

1. *The waves are propagated through the ether at the rate of 300,000,000 metres (186,300 miles) per second.*

As this is the known velocity of light, it must be the rate at which the waves are transmitted.

2. *The atoms of the ether vibrate at right angles to the line of the ray in all possible directions.*

It was at first assumed that the luminous vibrations, like the vibrations of sound, are *longitudinal*, that is, back and forth in the line of the ray; but the discoveries in polarization require that the vibrations of light should be assumed to be *transverse*, that is, in a plane perpendicular to the line of the ray; and, moreover, that in that plane the vibrations are in every possible direction within an inconceivably short space of time. Thus, if a person is looking at a star in the zenith, we must consider each atom of the ether between the star and his eye as vibrating across the vertical in all horizontal directions, north and south, east and west, and in innumerable lines between these.

3. *Different colors are caused by different rates of vibration.*

Red is caused by the *slowest* vibrations, and *violet* by the *quickest*, and other colors by intermediate rates. White light is to the eye what harmony is to the ear, the resultant effect of several rates of vibration combined. There are slower vibrations of the ether than those of red light, and quicker ones than those of violet light, but they are not adapted to affect the vision. The former affect the sense of feeling as *heat*, the latter produce chemical effects, and are called *actinic* rays.

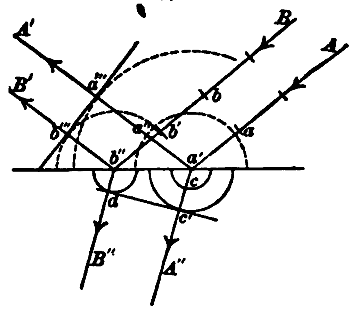
4. *The ether within bodies is less elastic than in free space.*

This is inferred from the fact that light moves with less velocity in passing through bodies than in free space; the greater the refractive power of a body, the slower does light move within it. And in some bodies of crystalline structure, it happens that the velocity is different in different directions, so that the elasticity of the ether within them must be regarded as varying with the direction.

423. Reflection and Refraction according to the Wave Theory.—The vibrations of the ether are transmitted from the source of motion as spherical waves. In a luminous body are an infinite number of radiants, each the centre of a succession of spherical waves. A beam of parallel rays is a collection of parallel radii of spherical waves, having different centres of disturbance, and the *wave front* of such a beam is the *tangent plane* common to all the spheres.

Suppose A and B to be two parallel rays of a beam of light. Let a and b (Fig. 267) represent two like wave fronts, and $a\ a' = b\ b'$ be the distance light moves in any small unit of time. When the wave a reaches a', b will have reached b'. While b' moves to b'', a' regarded as a centre of disturbance will have sent out a spherical wave to a''. While b'' is transmitting a spherical wave to b''', a'' will have extended to a''', and the common tangent plane $b'''\ a'''$ will be the wave front, and $A'\ B'$, the reflected rays, represent the beam. All rays from a' and b'' which move obliquely with respect to each other are destroyed by interference, only those remaining which move in parallel directions, as $A'\ B'$.

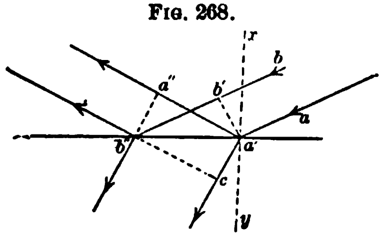
FIG. 267.



While b' moves to b'', a' is sending a wave into the medium at a slower rate, suppose with only half the velocity, which has advanced to c by the time b' reaches b''. While b'' is moving into the medium to d, c has moved to c', and a common tangent $d\ c'$ is the front of the refracted beam, of which $A''\ B''$ are rays.

424. Relation of Angles of Incidence, Reflection, and Refraction.—The velocity of light in any medium being uniform, retaining the notation of Fig. 267 we have in the triangles $a'\ b'\ b''$ and $a'\ a''\ b''$ (Fig. 268) $a'\ a'' = b'\ b''$, $a'\ b''$ common, and the angles at a'' and b' equal, being right angles formed by the radii and tangents; hence the angles $a''\ a'\ b''$ and $b'\ b''\ a'$ are equal; therefore the incident and reflected rays *must* make equal angles with the surface, and consequently with the normal.

FIG. 268.



In the triangle $a'\ b''\ b'$ we have $b'\ b'' = a'\ b'' \sin b'\ a'\ b''$, and in the triangle $a'\ b''\ c$ we have $a'\ c = a'\ b'' \sin a'\ b''\ c$.

$$\therefore \frac{b'\ b''}{a'\ c} = \frac{\text{sine } b'\ a'\ b''}{\text{sine } a'\ b''\ c} = \frac{\text{sine } x\ a'\ a}{\text{sine } c\ a'\ y} = \frac{\text{sine ang. Inc.}}{\text{sine ang. Refrac.}}.$$

But $b'\ b''$ and $a'\ c$ are spaces through which the wave is propa-

gated in the same interval of time, and as the velocities are constant in each medium, the ratio $\frac{b' \, b''}{a' \, c}$ must be a constant ratio; therefore its equal ratio $\frac{\text{sine ang. Inc.}}{\text{sine ang. Refrac.}}$ must also be a constant ratio.

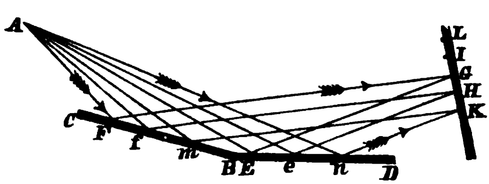
In the above it has been assumed that the velocity of transmission in the denser medium is less than in the rarer; this assumption is verified by direct experiment.

425. Interference.—As two systems of water-waves may increase or diminish their height by being combined, and as sounds, when blended, may produce various results, and even destroy each other, so may two pencils of light either augment or diminish each other's brightness, and even produce darkness.

If unlike phases coincide, the vibrations are destroyed and darkness follows, while if like phases meet, increase of brightness results.

To illustrate this, let two plane reflectors, inclined at a very obtuse angle, nearly 180°, receive light from a minute radiant, and reflect it to one spot on a screen; the reflected pencils will interfere, and produce bright and dark lines. Suppose light of one color, as violet, flows from a radiant point *A* (Fig. 269); let mirrors *B C* and *B D* reflect it to the screen *K L*. *F* and *E* may be so selected that the ray *A F* + *F G* equals the ray *A E* + *E G*.

FIG. 269.



Then *G* will be luminous, because the two paths being equal, the same phase of wave in each ray will occur at the point *G*. But if *H* be so situated that *A f* + *f H* differs *half a violet wave* from *A e* + *e H*, then *H* will be a dark point, because opposite phases meet there. A similar point, *I*, will lie on the other side of *G*. Again, there are two points, *K* and *L*, one on each side of *G*, to each of which the whole path of light by one mirror will exceed the whole by the other by just *one* violet wave; those points are bright.

If the paths differ by any odd multiple of ½ wave length, light is destroyed and a dark band is seen; but if these paths differ by

any multiple of a whole wave length, the light is intensified and bright bands are seen.

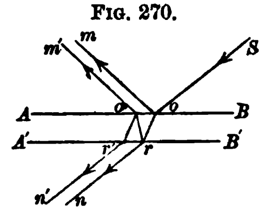
Thus, there is a series of bright and dark points on the screen; or rather a series of bright and dark hyperbolic *lines*, of which these points are sections. Other colors will give bands separated a little further, indicating longer waves. And white light, producing all these results at once, will give a repetition of the prismatic series.

426. Striated Surfaces.—If the surface of any substance is ruled with fine parallel grooves, 2000 or more to the inch, these grooves will act like the inclined mirrors of the last paragraph, and it will reflect bright colors when placed in the sunbeam. *Mother-of-pearl* and many kinds of sea-shell exhibit colors on account of delicate striæ on their surface. It may be known that the color arises from such a cause, if, when the substance is impressed on fine cement, its colors are communicated to the cement. Indeed, it was in this way that Dr. Wollaston accidentally discovered the true cause of such colors. The changeable hues in the plumage of some birds, and the wings of some insects, are owing to a striated structure of their surfaces. Professor Rowland has succeeded in ruling 43000 lines to the inch on speculum metal, this, too, on a concave surface.

Gilt buttons and other articles for dress are sometimes stamped with a ruling die, and are called *iris ornaments.* The color in a given case depends on the distance between the grooves, and the obliquity of the beam of light. Hence, the same surface, uniformly striated, may reflect all the colors, and every color many times, by a mere change in its inclination to the beam of light.

427. Thin Laminæ.—Thin films of transparent substances, such as soap-bubbles, thin blown glass, oil on water, present a colored appearance to an observer viewing them by reflected or transmitted light. The color varies with changes in the thickness of the film. The phenomena is one of interference. To understand it let us consider $A\ B$ and $A'\ B'$ (Fig. 270) to be the bounding surfaces of a thin film of glass, surrounded by air. Let a ray of red light come from S and strike the surface $A\ B$ at o. A portion of it will be reflected to m, and the rest refracted to r, where still another portion is internally reflected to o' and then refracted to m'. Now, as the film is very thin, $o\ m$ and $o'\ m'$ practically coincide, *i.e.*, o and o' are coincident, and both rays would enter the eye of an

FIG. 270.



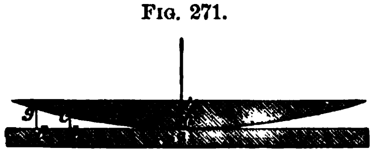
observer. At the point r of internal reflection the ray loses half a wave-length (Art. 293). This always happens when reflection takes place at the surface of a rarer medium. Now, if the path $o\ r\ o'$ (dependent on the thickness of the film) is an *even* number of *half wave lengths* of red light, the ray emerging from o' will be opposite in phase to that reflected at o. If the intensities of the two rays were the same, darkness would result from the interference. For instance, suppose $o\ r\ o'$ was equal to four half wave lengths, *i.e.*, two wave lengths; adding the half wave lost at r we have light issuing from o' two and a half wave lengths behind o. It is opposite in phase and darkness results. On the other hand, if the path $o\ r\ o'$ is an *odd* number of half-wave lengths, o and o' are in the same phase, and increase of brightness results. If the observer moves his eye in a direction parallel to the surface of the film, he sees brightness and darkness alternately, the obliquity of the rays altering the length of the path in the film.

When S is a source of white light, colors result. For, in a given position of the observer, red waves may be quenched while others are strengthened in a more or less degree.

If we examine the *transmitted* rays $S\ n$ and $S\ r\ o'\ n'$, the latter having been twice reflected internally, we find, by the same reasoning, that the thickness which would produce opposite phases by reflection now produces like phases. Colors by reflection are supplanted by their complementary colors in transmission.

428. Newton's Rings.—If a lens of slight convexity is laid on a plane lens, and the two are pressed together by a screw, and viewed by reflected light, rings of color are seen arranged around the point of contact. The rings of least diameter are broadest and most brilliant, and each one contains the colors of the spectrum in their order, from violet on the inner edge to red on the outer. But the larger rings not only become narrower and paler, but contain fewer colors; yet the succession is always in the same order as above. Viewed by monochromatic light they appear as successive rings of brightness and darkness. They are caused by the interference between rays reflected from the upper air surface between the lenses (with loss of half a wave length) and rays reflected from the lower lens. A given color appears in a circle around the point of contact, because the circle marks the places where the film is of equal thickness. *Measurements of the diameters of the successive rings of different colors show that their squares are as the odd num-*

FIG. 271.



bers, 1, 3, 5, 7; and hence the thicknesses of the laminæ of air at the repetitions of the same color are as the same numbers. For, let Fig. 271 represent a section of the spherical and plane surfaces in contact at a. Let $a\,b$, $a\,d$, be the radii of two rings at their brightest points. Suppose $a\,i$, perpendicular to $m\,n$, to be produced till it meets the opposite point of the circle of which $a\,g$ is an arc, and call that point f; then $a\,f$ is the diameter of the sphere of which the lens is a segment. Let $b\,e$, $d\,g$, be parallel to $a\,i$, and $e\,h$, $g\,i$, to $m\,n$, then we have

$$(e\,h)^2 : (g\,i)^2 :: a\,h \times h\,f : a\,i \times i\,f.$$

But the distances between the two lenses being exceedingly small in comparison with the diameter of the sphere, $h\,f$ and $i\,f$ may be taken as equal to $a\,f$; whence, by substitution,

$$(e\,h)^2 : (g\,i)^2 :: a\,h \times a\,f : a\,i \times a\,f :: a\,h : a\,i :: b\,e : d\,g.$$

But, as has been stated, careful measurement of the diameters (or radii) has shown that

$$(e\,h)^2 : (g\,i)^2 :: 1 : 3.$$

Hence we have $\quad b\,e : d\,g :: 1 : 3.$

Therefore the thicknesses of successive rings are as the odd numbers.

429. Thickness of Laminæ for Newton's Rings.—The absolute thickness, $b\,e$, $d\,g$, &c., can also be obtained, $a\,f$ being known, since

$$a\,f : a\,e :: a\,e : a\,h \text{ or } b\,e;$$

for in so short arcs the chord may be considered equal to the sine, that is, the radius of the ring.

With air between the lenses Newton found the thickness of the first bright ring of orange-yellow light to be $\frac{1}{178000}$ of an inch. This distance being twice traversed by one ray, and there being a loss of half a wave at the point of reflection of the other ray, we have $\frac{2}{178000}$ = half a wave length l.

$$\therefore l = \tfrac{4}{178000} = .000022.$$

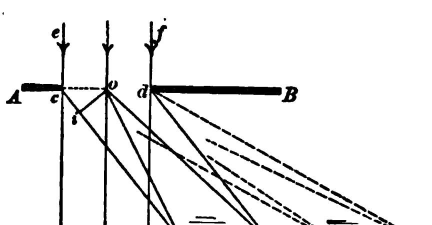
When air is between the lenses, all the rings range between the thickness of *half a millionth* of an inch and 72 *millionths*; if water is used, the limits are $\frac{3}{8}$ *of a millionth* and 58 *millionths*. Below the smaller limit the medium appears black, or no color is reflected; above the highest limit the medium appears white, all colors being reflected together. When water is substituted for air, all the rings contract in diameter, indicating that a particular order of color requires less thickness of water than of air; the thicknesses for different media are found to be in the inverse ratio of the indices of refraction.

430. Relation of Rings by Reflection and by Transmission.—If the eye is placed beyond the lenses, the transmitted light also is seen to be arranged in very faint rings, the brightest portions being at the same thicknesses as the darkest ones by reflection; and these thicknesses are as the even numbers, 2, 4, 6, &c. The centre, when black by reflection, is white by transmission, and where red appears on one side, blue is seen on the other; and, in like manner, each color by reflection answers to its *complementary color* by transmission, according to Article 427.

431. Newton's Rings by a Monochromatic Lamp.—The number of reflected rings seen in common light is not usually greater than from *five* to *ten*. The number is thus small, because as the outer rings grow narrower by a more rapid separation of the surfaces, the different colors overlap each other, and produce whiteness. But if a light of only one color falls on the lenses, the number may be multiplied to several hundreds; the rings are alternately of that color and black, growing more and more narrow at greater distances, till they can be traced only by a microscope. A good light for such a purpose is the flame of an alcohol lamp, whose wick has been soaked in strong brine, and dried.

432. Diffraction or Inflection.—One of the forms of inflection is explained as follows: Through an opaque screen, *A B* (Fig. 272), let there be a very narrow aperture, *c d*, by which is admitted the beam of red light, *e f g h*, emanating from a single point.

FIG. 272.



All points of the wave front *c d* are radiants, from which emanate waves in all directions. The original wave will move on through the aperture forming the band of light *g a h*. A ray from *c* as a radiant will interfere with that ray from *o*, which is one-half wave length in advance of it, and will give darkness at some point *j*; all rays parallel to *c j* emanating from points between *c* and *o*, will interfere with corresponding rays from points between *o* and *d*, parallel to *o j*, and the total result will be the dark band *j k*.

In like manner rays from those points which differ by $\frac{3}{2}$ wave lengths will interfere and produce the second dark band *m n*. Between these will be a bright band, *k m*, due to waves which differ by one whole wave length. The obliquity of the rays in the

figure has been very greatly exaggerated, that the lines may not be confused. The general plan of the determination of the wave length of the color used may be made plain by reference to the figure. Let $o\,i$ be drawn perpendicular to $c\,j$. Then in the triangle $c\,o\,i$ we have $c\,i = c\,o \times \cos o\,c\,i = \frac{1}{2}\,c\,d \times \text{sine}\,j\,c\,g$.

Now, because the screen is at a great distance from the slit as compared with $c\,d$, which is about $\frac{1}{16}$ of an inch, and as $g\,c\,j$ is very small indeed, about 1.5′, we have $j\,o = j\,i$, and hence $c\,j - o\,j = o\,j - i\,j = c\,i$; therefore, in order to find $c\,i$ ($= \frac{1}{2}$ a wave length) we must measure $c\,d$ and the angular deviation $g\,c\,j$. Instead of a single aperture, $c\,d$, a great number of very fine parallel lines ruled on glass are used, and details of measurement are adopted, a description of which is beyond our limits. With white light, prismatic fringes would be produced.

433. Inflection by One Edge of an Opaque Body.—If one side of the aperture $c\,d$ in the last paragraph be removed, the effect, while due to the same cause, will be somewhat modified. Let a convex lens converge sunlight to a focus from which it again diverges, the room being dark. If we introduce into the divergent pencil any opaque body, as a knife-blade, for example, and observe the shadow which it casts on a white screen, we shall observe on both sides of the shadow *fringes of colored light*, the different colors succeeding each other in the order of the spectrum, from violet to red. Three or four series can usually be discerned, the one nearest to the shadow being the most complete and distinct, and the remoter ones having fewer and fainter colors. The phenomenon is independent of the density or thickness of the body which casts the shadow. The light, in passing by the edge or back of a knife, by a block of marble or a bubble of air in glass, is in each case affected in the same way. But if the body is very narrow, as, for example, a fine wire, a modification arises from the light which passes the opposite side; for now fringes appear *within* the shadow, and at a certain distance of the screen the central line of the shadow is the most luminous part of it.

If light of one color be used, and the distance of the color from the edge of the shadow be measured when the screen is placed at different distances from the body, it will be found that the distances from the shadow are not proportional to the distances of the screen from the body; which proves that the color is not propagated in a straight line, but in a curve. These curves are found to be *hyperbolas*, having their concavity on the side next the shadow, and are in fact a species of caustics.

434. Light through Small Apertures.—The phenomena of inflection are exhibited in a more interesting manner when we

view with a magnifying glass a pencil of light after it has passed through a small aperture. For instance, in the cone already described as radiating from the focus of a lens in a dark room, let a plate of lead be interposed, having a pin-hole pierced through it, and let the slender pencil of light which passes through the pin-hole fall on the magnifier. The aperture will be seen as a luminous circle surrounded by several rings, each consisting of a prismatic series. These are, in truth, the fringes formed by the edge of the circular puncture, but they are modified by the circumstance that the opposite edges are so near to each other. If, now, the plate be removed, and another interposed having *two* pin-holes, within one-eighth of an inch of each other, besides the colored rings round each, there is the additional phenomenon of long lines crossing the space between the apertures; the lines are nearly straight, and alternately luminous and dark, and varying in color, according to their distance from the central one. These lines are wholly due to the overlapping of two pencils of light, for on covering one of the apertures they entirely disappear. By combining circular apertures and narrow slits in various patterns in the screen of lead, very brilliant and beautiful effects are produced.

435. Why Inflection is not Always Noticed in Looking by the Edges of Bodies.—It must be understood that light is *always* inflected when it passes by the edges of bodies; but that it is rarely observed, because, as light comes from various sources at once, the colors of each pencil are overlapped and reduced to whiteness by those of all the others. By using care to admit into the eye only isolated pencils of light, some cases of inflection may be observed which require no apparatus. If a person standing at some distance from a window holds close to his eye a book or other object having a straight edge, and passes it along so as to come into apparent coincidence with the sash-bars of the window, he will notice, when the edge of the book and the bar are very nearly in a range, that the latter is bordered with colors, the violet extremity of the spectrum being on the side of the bar nearest to the book, and the red extremity on the other side. Again, the effect produced when light passes through a narrow aperture may be seen by looking at a distant lamp through the space between the bars of a pocket-rule, or between any two straight edges brought almost into contact. On each side of the lamp are seen several images of it, growing fainter with increased distance, and finely colored. An experiment still more interesting is to look at a distant lamp through the net-work of a bird's feather. There are several series of colored images, having a fixed arrange-

ment in relation to the disposition of the minute apertures in the feather; for the system of images revolves just as the feather itself is revolved.

436. Length and Number of Luminous Waves.—The careful measurements which have been made in cases of interference have led, by many independent methods, to the accurate determination of the length of a wave of each color. When the length of a wave of any color is known, the number of vibrations per second is readily obtained by dividing the velocity of light by the length of the wave, for light moves a distance equal to one wave length during one vibration (Art. 421); therefore if we divide the distance per second, 300,000,000 meters, by the length of one wave, we have the number of vibrations per second as above.

The results of these investigations give for the

	Frequencies.	Wave lengths in cms.
Red (Line A)	395,000,000,000,000	0.00007604
Yellow (Line D^1)	508,905,810,000,000	0.00005895
Violet (Line H_2).........	763,600,000,000,000	0.00003933

437. Calorescence and Fluorescence.—Rays of less refrangibility than the extreme red are due to vibrations too slow to effect vision, but by their great number they possess great heating power. If a beam of light be allowed to fall upon a thin layer of a solution of iodine contained in a suitable cell, all light rays will be absorbed, and nearly all the rays of slow vibration will pass through. These, if brought to a focus, will communicate to refractory substances vibrations sufficiently rapid to be recognized by the organ of vision. The phenomenon is termed Calorescence.

Other rays, of vibrations too rapid to be recognized as light, are also found in the spectrum far beyond the violet. If these are allowed to fall upon a solution of sulphate of quinine, or upon paper impregnated with æsculine, or upon other substances capable of reducing the rate of vibration, these substances become visible, glowing with a color peculiar to each solution, and determined by the rate of vibration which has resulted. This property of substances by which the ultra violet rays are made visible is called Fluorescence.

438. Phosphorescence.—Very many substances. if they are exposed to a strong light and then are transferred to a dark chamber, continue to emit light for longer or shorter periods, depending upon the substance used. The sulphides of calcium

and strontium remain luminous for hours after exposure to sunlight.

Many other substances possess the property of phosphorescence in so slight a degree, that they will emit light for only a fraction of a second after being withdrawn from the sun's rays, while many seem not to possess it at all.

What is called phosphorescence in certain animals and in decaying animal and vegetable substances has no relation to that just described, and can not properly be considered here.

CHAPTER VII.

DOUBLE REFRACTION AND POLARIZATION.

439. Change of Vibrations in Polarized Light.—It has been stated (Art. 422) that the vibrations of the ether in the case of common light, must be supposed to be *transverse in all directions.* But, instead of this, we may conceive, what is mechanically equivalent to it, that the vibrations are made in *two* transverse directions at right angles to each other, and to the direction of the ray.

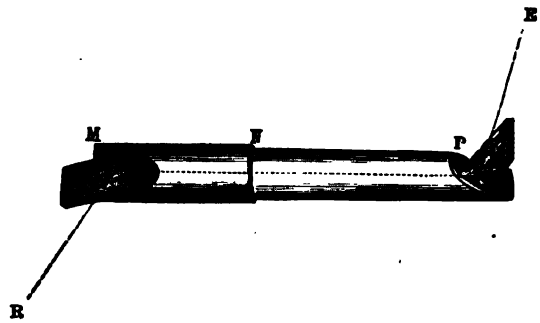
This being the nature of common light, it is easy to state what is meant by polarized light. It is that in which the vibrations are performed in only *one* of the transverse directions. It is, of course, immaterial what particular transverse motion is cut off, provided all the motion at right angles to it is retained.

440. Polarizing and Analyzing by Reflection.—When light is reflected, those vibrations of the ray which are *in* the plane of incidence are generally weakened in a greater or less degree, while those which are *perpendicular* to the same plane are not affected. How much the vibrations are weakened depends on the elasticity of the ether within the medium, and on the angle of incidence. But reflection of light rarely if ever takes place without diminishing the amplitude of those vibrations which are in the plane of incidence; so that a reflected ray is always polarized, at least, in a slight degree.

441. Polarization by Reflection.—Let two tubes, *M N* and *N P* (Fig. 273) be fitted together in such a manner that one can be revolved upon the other; and to the end of each let there be attached a plate of dark-colored glass, *A* and *C*, capable of reflect-

ing only from the first surface. These plates are hinged so as to be adjusted at any angle with the axis of the tube. Let the plane of each glass incline to the axis of the tube at an angle of 33°, and let the beam *R A* make an incidence of 57°, the complement

FIG. 273.



of 33°, on *A*; then it will, after reflection, pass along the axis of the tube, and make the same angle of incidence on *C*. If now the tube *N P* be revolved, the second reflected ray will vary its intensity, according to the angle between the two planes of incidence on *A* and *C*. The beam *A C* is *polarized light;* the glass *A*, which has produced the polarization, is called the *polarizing plate;* the glass *C*, which shows, by the effects of its revolution, that *A C* is polarized, is *the analyzing plate;* and the whole instrument, constructed as here represented, or in any other manner for the same purpose, is called a *polariscope.*

442. Changes of Intensity Described.—The changes in the ray *C E* are as follows:

Since all vibrations except those perpendicular to the plane of incidence have been destroyed, when the tube *N P* is placed so that the plane of incidence on *C* is coincident with the former plane of incidence, *R A C*, whether *C E* is reflected forward or backward in that plane, the intensity at *E* will be the same as if *A C* had been a beam of common light. If *N P* is revolved, *E* will begin to grow fainter, and reach its minimum of intensity when the planes *R A C* and *A C E* are at right angles, which is the position indicated in the figure; for only vibrations at right angles to the plane of incidence can be reflected, and there are no such vibrations with reference to this second plane of incidence.

Continuing the revolution, we find the intensity increasing through the second quadrant of revolution, and reaching its maximum, when the two planes of incidence again coincide, 180° from the first position.

No reflection polarizes perfectly, and hence there will be increase and decrease of the intensity of the reflected ray, without total extinction.

443. The Polarizing Angle.—The angle of 57° is called the polarizing angle for glass, not because glass will not polarize at other angles of incidence, but because at all other angles it polarizes the light in a less degree; and this is indicated by the fact that, in revolving the analyzing plate, there is less change of intensity, and the light at E does not become so faint. Different substances have different polarizing angles, and for that angle of incidence for any substance which will produce a maximum of polarization, the *reflected* and *refracted* rays will make with each other an angle of 90°. Hence the refractive power of opaque bodies may be determined. The polarization produced by reflection from the metals is very slight.

444. Polarization by a Bundle of Plates. — Light may also be polarized by *transmission* through a bundle of laminæ of a transparent substance, at an angle of incidence equal to its polarizing angle.

Since the *reflected* ray in perfectly polarized light consists of vibrations only at right angles to the plane of incidence, the *transmitted* light, being deprived of these vibrations, will consist of vibrations only *in* the plane of incidence and refraction. As no single reflection perfectly sifts out the vibrations at right angles to the planes of incidence and reflection, many reflections at successive surfaces are secured, so that finally there will remain in the *refracted* ray only vibrations in the plane of incidence and refraction.

Let a pile of twenty or thirty plates of transparent glass, no matter how thin, be placed in the same position as the reflector A, in Fig. 273, and a beam of light be transmitted through them in a direction toward C. In entering and leaving the bundle A, situated as in the figure, the angles of incidence and refraction are in a horizontal plane. When C is revolved, the beam undergoes the same changes as before, with this difference, that the places of greatest and least intensity will be reversed. If the light is reflected from C in the same plane in which it was refracted by A, its intensity is least, and it is greatest when reflected in a plane at right angles to it, as at E in the figure.

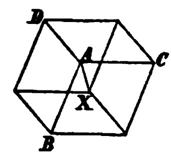
445. Polarization by Absorption.—The third and most perfect method of polarizing light, is by *transmission through certain crystals.* Some crystals polarize the transmitted light by *absorption.* If a thin plate be cut from a crystal of tour-

maline, by planes parallel to its axis, the beam transmitted through it is polarized, the vibrations parallel to the axis being transmitted and those perpendicular to the axis being absorbed, and, when received on the analyzing plate, will alternately become bright and faint, as the tube of the analyzer is revolved.

If the analyzer be a plate of tourmaline similar to the polarizer the rays will pass when the axes of the plates are parallel, but will be wholly absorbed when the axes are at right angles to each other.

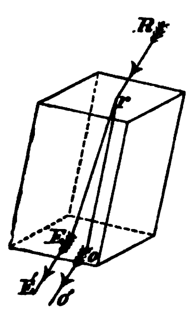
446. Double Refraction.—There are many transparent substances, particularly those of a crystalline structure, which, instead of refracting a beam of light in the ordinary mode, *divide it into two beams.* This effect is called *double refraction,* and substances which produce it are called *doubly-refracting substances.*

FIG. 274.



This phenomenon was first observed in a crystal of carbonate of lime, denominated *Iceland spar.* It is bounded by six rhomboidal faces, whose inclinations to each other are either 105° 5′, or 74° 55′. There are two opposite solid angles, A and X (Fig. 274), each of which is formed by the meeting of three obtuse plane angles. A line drawn in such direction as to be equally inclined to the three edges of either of these obtuse solid angles, is called the axis of the crystal. If the edges of the crystal were equal, the axis would be the diagonal A X of the rhomb.

FIG. 275.



A ray of light passing through the crystal in the direction of the axis does not suffer double refraction ; passing in any other direction, however, it does.

The elasticity of the ether in the crystal is different in the direction of the axis from what it is in a direction at right angles to this axis. Now, remembering that the velocity of propagation depends upon the elasticity of the medium (Art. 422), and that the velocity determines the refractive index (Art. 424), we can see that the two sets of vibrations of a ray of light R (Fig. 275), which take place in rectangular planes, will encounter different elasticities upon entering a crystal at r. They will accordingly have different refractive indices and will separate into two rays.

447. Ordinary and Extraordinary Rays.—Any plane which contains the optical axis of a crystal is termed a *principal*

section. Place a principal section of a crystal upon a black dot on a sheet of paper. The dot will appear as two, owing to the double refraction. Revolve the plate around an axis perpendicular to the paper. One of the images will remain stationary while the other revolves about it. The rays which form the stationary image will be found to conform to the ordinary laws of refraction (Art. 369), and are termed the *ordinary rays*. The rays which form the movable image will be found to depart from these laws. Either the refractive index will not be constant, or the refracted ray will not be in the plane of incidence. Both these discrepancies may occur. These rays are called *extraordinary rays*.

The refractive index of the extraordinary ray may be greater or smaller than that of the ordinary ray, depending upon the refracting substance. Crystals of a *positive* axis, are those in which the extraordinary ray has a *larger* index of refraction than the ordinary ray; crystals of a *negative* axis are those in which the index of the extraordinary ray is less than that of the ordinary ray. Iceland spar is a crystal of negative axis.

Some crystals have two axes of double refraction; that is, there are two directions in which light may be transmitted without being doubly refracted. A few crystals have more than two axes.

In the experiment mentioned above we may assume that the crystal selects those vibrations which are in the direction of the optical axis to form the ordinary image, and those at right angles to it to form the extraordinary image.

The property of double refraction belongs to a large number of crystals, and also to some animal substances, as hair, quills, &c.; and it may be produced artificially in glass by heat or pressure.

448. Polarizing by Double Refraction.—If a beam is passed through a doubly-refracting crystal, and the two parts fall on the analyzing plate, they will come to their points of greatest and least brightness at alternate quadrants; indeed, when one ray is brightest, the other is entirely extinguished. Therefore the two rays which emerge from a doubly-refracting crystal are polarized *completely*, the ordinary ray in a principal plane and the extraordinary ray in a plane at right angles to a principal plane.

449. Different Kinds of Polarization.—Since the discovery was made that the ethereal atoms may by certain methods be thrown into circular movements, and by others into vibrations in an ellipse with the axis in a fixed direction, the polarization already described has been called *plane polarization*, since the atoms vibrate in a plane. *Circular polarization* is that in which the atoms revolve in circles; and *elliptical polarization* denotes a

state of vibration in ellipses, whose major axes are confined to one plane.

The consideration of these various modes of polarization demands more space than can be spared here.

450. Every Polarizer an Analyzer.—We have seen that light is polarized by reflection from glass at an incidence of 57°, and analyzed by another plate at the same angle of incidence. This is but an instance of what is always true, that every method of polarizing light may be used to analyze, *i.e.*, to test its polarization. Hence, a bundle of thin plates of glass may take the place of the analyzer *C*, as well as of the polarizer *A*. For, on turning it round, though the transmitted beam remains in the same place, yet it will, at the alternate quadrants, brighten to its maximum and fade to its minimum of intensity.

So, again, if light has passed through a tourmaline, and is received on a second whose crystalline axis is parallel to that of the former, the ray will proceed through that also; but if the second is turned in its own plane, the transmitted ray grows faint, and nearly disappears at the moment when the two axes are at 90° of inclination, and this alternation continues at each 90° of the whole revolution.

Finally, place a double-refractor at each end of the polariscope, and let a beam pass through them and fall on a screen. The first crystal will polarize each ray, and the second will doubly refract and also analyze each, exhibiting a very interesting series of changes. In general, four rays will emerge from the second crystal, producing four luminous spots on the screen. But, on revolving the tube, not only do the rays commence a revolution round each other, but two of them increase in brightness, and the other two at the same time diminish as fast, till two alone are visible, at their greatest intensity. At the end of the second quadrant, the spots before invisible are at their maximum of brightness, and the others are extinguished. This alternation continues as long as the crystal is revolved. In the middle of each quadrant the four are of equal brightness.

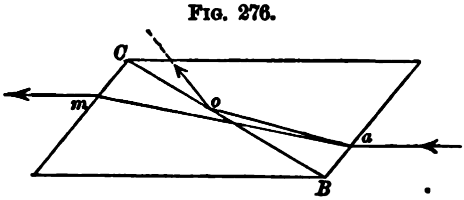
The separation of a plane polarized ray into two plane polarized rays by a double refracting substance can happen only when the original plane coincides with neither of the emergent planes. The double refracting substance resolves the incident vibration into two component vibrations at right angles to each other. In case the original plane coincides with one of the double refracting crystal's chosen planes the brightness in the other plane will be nil.

451. Nicol's Prism.—As the four beams in the last case are

an annoyance in investigations requiring polarized light, only one is retained, the others being turned aside.

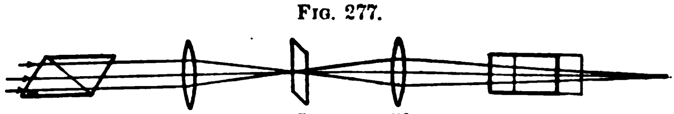
A rhomb of Iceland spar is cut by a plane $A\ B\ X$ passing through the obtuse angles (Fig. 274); the two halves are then joined together again, as they were before the cutting, by Canada balsam. This is a Nicol prism. If a beam enters the prism at a (Fig. 276), it will be separated into two beams, the ordinary and the extraordinary, and then will fall upon the film of Canada balsam $C\ B$. Now the refractive index of Canada balsam is less than the index of refraction for the ordinary ray, and greater than that for the extraordinary ray; hence the ordinary ray will be totally reflected at o, and will pass out at the side of the prism, while the extraordinary ray will be refracted through the film of balsam, emerging as polarized light at m. If a similar prism be used as an analyzer, one of the two rays into which the polarized ray is separated is again turned out of the prism by total reflection, and only a single ray emerges.

FIG. 276.



452. Color by Polarized Light.—Suppose that apparatus is arranged as in Fig. 277, N being a Nicol's prism for a polarizer,

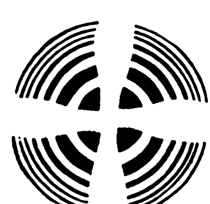
FIG. 277.



L being a lens which converges parallel rays from the polarizer to C, which is a section of a double refracting crystal cut perpendicular

FIG. 278 (*a*).

FIG. 278 (*b*).



to the optic axis, L' being another lens which brings the rays to a focus beyond the analyzing Nicol's prism N'. If, now, monochro-

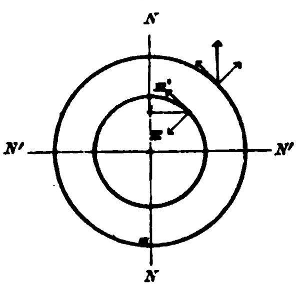
matic, *e.g.*, red light, be used, and if the polarizing planes of N and N' be perpendicular to each other, the crystal being removed, no light will be perceived by an eye placed beyond N'. If the crystal be now inserted, the eye sees a series of red and black rings, covered by a black cross, as shown in Fig. 278 (a). Upon changing to white light, the cross remains black, but the rings assume prismatic colors. By turning N' so that its polarizing plane coincides with that of N, the dark cross will change to a white one and the colored rings will change to their complementary colors. This is indicated in Fig. 278 (b).

In order to explain these phenomena, let us start with the polarizer and analyzer crossed, and with monochromatic light. Any vibrations from N, which are not altered by the crystal, will fail to pass through N' and will give an eye the impression of black. The rays through the optic axis are not modified, and hence the black centre. Rays striking the crystal in any other direction than that of the axis will be doubly refracted, splitting up into two rays with vibrations in planes perpendicular to each other. One of these planes will pass through the axis and the other will be parallel to the axis, but perpendicular to the first plane. The student may imagine that a uniaxial doubly refracting crystal is constructed like the trunk of a tree. The axis corresponds to the central heart of the tree, and this is surrounded by rings of the material. The elasticity along the grain (and between the rings) is different from that across the grain. Now, suppose vibrations in a plane to emanate from a point on the axis produced. Those passing along the axis will be unaltered. Those striking the outside rings of the crystal at points lying in the plane of polarization passing through the axis will be singly refracted, having zero components in the direction of the tangents to the rings at these points. The analyzer will thus cause one arm of the cross (in this plane) to be dark. In the plane passing through the axis and perpendicular to the one just mentioned, all the vibrations will be in the directions of the tangents, and the component along the radius will be zero. The analyzer will thus cause the other arm of the cross to be dark. At all other points the vibrations will have definite components both in the tangental and the radial directions. The two resulting rays will travel with different velocities, because one has to vibrate across the grain toward the centre and the other vibrates with the *grain between the concentric layers.* They may thus emerge from the crystal in different phases. Whether they are in like or opposite phases depends upon the distance they have had to travel in the crystal. This distance depends upon the obliquity with which they strike the crystal, *i.e.*, upon the distance of the incident point

from the central axis. From equal distances from the centre, *i.e.*, from the circumference of a circle, they will emerge in like phase; from another circle they will emerge in opposite phase. The ether particles on the emergent side of the crystal will move under the influence of both systems of vibrations. They will describe resultant paths. These paths will be in the plane of N''s polarization or perpendicular to it, according as the component vibrations are in like or opposite phase. The impressions on the eye of an observer will be red or black respectively.

The composition of the component vibrations and the function of the analyzer can be seen in Fig. 279. Let the Figure represent the surface of the crystal towards N'. Let the inner circle represent the locus of points where the emergent vibrations are in like phase and the outer circle the locus of those in opposite phase. $N' N'$ is the plane of polarization of the analyzer, $N N$ that of the polarizer. An ether particle on the inner circle 45° from $N N$ will be subjected to two impulses E and E', perpendicular to each other. These will be equal to each other, for at this angle a ray from the polarizer is divided into two equally intense rays. The resulting path of the particle will be the diagonal of the parallelogram. This corresponds with the analyzer's plane of polarization and the ray passes through unextinguished. On the outer circle the direction of one vibration must be changed, for here the vibrations are in unlike phase. Resolving, we get a vibration at right angles to the plane of the analyzer. The ray will, therefore, be extinguished by the analyzer.

FIG. 279.



It can be readily seen that white light will cause prismatically colored rings (the cross remaining black), as was explained in Art. 427.

Making the polarization planes of N and N' coincident will, of course, change the black cross to a white one, and will change colors into their complementaries.

453. Rotation of the Plane of Polarization.—Some substances have the property of twisting the plane of polarization of light while it is traversing them. For instance, if a solution of sugar be inserted between an analyzer and polarizer, the former of which has been previously placed so as to extinguish the light from the polarizer, the light will reappear. In order to again produce extinction, the analyzer must be rotated through a certain

angle. For a given sugar the magnitude of this angle varies as the concentration of the solution and as the length of it traversed by the light. Some substances turn the plane to the right, others to the left. This peculiarity of sugar solutions is made use of by the Custom-house Department in appraising the value of syrups, etc.

CHAPTER VIII.

VISION.

454. Image by Light through an Aperture.—If light from an external object pass through a small opening of any shape in the wall of a dark room, it will form an ill-defined inverted image on the opposite wall. Imagine a minute square orifice, through which the light enters and falls on a screen several feet distant. A pencil of light, in the shape of a square pyramid, emanating from the highest point of the object, passes through the aperture, and forms a luminous square near the bottom of the screen. From an adjacent point another pencil, crossing the first at the aperture, forms another square, overlapping and nearly coinciding with the former. Thus every point of the object is represented by its square on the screen; and as the pencils all cross at the aperture, the image formed is every way inverted. It is also indistinct, because the squares overlap, and the light of contiguous points is mingled together. If the orifice is smaller, the image is less luminous, but more distinct, because the pencils which form it overlap in a less degree. If the hole is circular, or triangular, or of irregular form, there is no change in the appearance of the image, which is now composed of small circles, or triangles, or irregular figures, whose shape is completely lost by overlapping.

455. Effect of a Convex Lens at the Aperture.—The image will become distinct, and more luminous also, if the aperture be enlarged to a diameter of two or three inches, and then covered by a convex lens of the proper curvature. The image will be *distinct*, because the rays from each point of the object are converged to a point again, and *luminous* in proportion as the lens has a larger area than the aperture before employed. This is a real, and therefore an inverted image (Art. 385). A *scioptic ball* is a sphere containing a lens, and so fitted in a socket that it can be turned in any direction, and thus bring into the room the

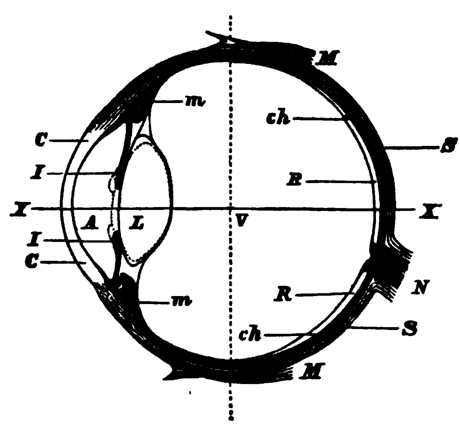
images of different parts of the landscape. The *camera obscura* is a darkened room furnished with a scioptic ball and adjustable screen for producing distinct pictures of external objects.

Instead of connecting the lens with the wall of a room, it is frequently attached to a portable box or case, within which the image is formed. The *Daguerreotype*, or *photograph*, is the image produced by the convex lens, and rendered permanent by the chemical action of light on a surface properly prepared. The lens for photographic purposes needs to be achromatic, and corrected, also, as far as possible, for spherical aberration.

456. The Eye.—The eye is a camera obscura in miniature; we find here the darkened room, the aperture, the convex lens, and the screen, with inverted images of external objects projected on it. A horizontal section of the eye is represented in Fig. 280.

The optical apparatus of the eye, and the spherical case which incloses it, constitute what is called the *eye-ball*. The case itself, except about a sixth part of it in front, is a strong white substance, called, on account of its hardness, the *sclerotic coat*, *S*, *S* (Fig. 280). In the front, this opaque coat changes to a perfectly transparent covering, called the *cornea*, *C*, *C*, which is a little more convex than the sclerotic coat. The increased convexity of the cornea may be felt by laying the finger gently on the eyelid when closed, and then rolling the eye one way and the other.

Fig. 280



The bony socket, which contains the eye, is of pyramidal form, its vertex being some distance behind the eyeball; room is thus afforded for the mechanism which gives it motion. This cavity, except the hemisphere in front occupied by the eye itself, is filled up with fatty matter and with the six muscles by which the eyeball is revolved in all directions.

457. The Interior of the Eye.—Behind the cornea is a fluid, *A*, called the *aqueous humor*. In the back part of this fluid lies the *iris*, *I*, *I*, an opaque membrane, having in the centre of it a circular aperture, the *pupil*, through which the light enters.

The iris is the colored part of the eye; the back side of it is black. Directly back of the aqueous humor and iris, is a flexible double convex lens, L, called the *crystalline lens*, or *crystalline humor*, having the greatest convexity on the back side. The large space back of the crystalline is occupied by the vitreous humor, V, a semi-liquid, of jelly-like consistency. Next to the vitreous humor succeed those inner coatings of the eye, which are most immediately concerned in vision. First in order is the *retina*, R, R, on which the light paints the inverted pictures of external objects. The fibres of the optic nerve, which enter the ball at N, are spread all over the retina, and convey the impressions produced there to the brain. Outside of the retina is the *choroid coat*, $c\,h$, $c\,h$, covered with a black pigment, which serves to absorb all the light so soon as it has passed through the retina and left its impressions. The choroid is inclosed by the sclerotic already described. The nerve-fibres, which are spread over the interior of the retina, are gathered into a compact bundle about one-tenth of an inch in diameter, which passes out through the three coatings at the back part of the ball, about fifteen degrees from the axis, $X\,X$, on the side toward the other eye. M, M represent two of the muscles, where they are attached to the eyeball.

458. Vision.—The index of refraction for the cornea, and the aqueous and vitreous humors, is just about the same as that for water; for the crystalline lens, the index is a little greater. The light, therefore, which comes from without, is converged principally on entering the cornea, and this convergency is a little increased both on entering and leaving the crystalline. If the convergency is just sufficient to bring the rays of each pencil to a focus on the retina, then the images are perfectly formed, and there is distinct vision. To prevent the reflection of rays back and forth within the chamber of the eye, its walls are made perfectly black throughout by a pigment which lines the choroid, the ciliary processes, and the back of the iris. Telescopes and other optical instruments are painted black in the interior for a similar purpose.

The cornea is prevented from producing spherical aberration by the form of a prolate spheroid which is given to its surface, and the crystalline, by a gradual increase of density from its edge to its centre.

459. Adaptations.—By the prominence of the cornea rays of considerable obliquity are converged into the pupil, so that the eye, without being turned, has a range of vision more or less perfect, through an angle of about 150°.

The quantity of light admitted into the eye is regulated by the size of the pupil. The iris, composed of a system of circular and

radial muscles, expands or contracts the pupil according to the intensity of the light. These changes are involuntary; a person may see them in his own eyes by shading them, and again letting a strong light fall upon them, while he is before a mirror.

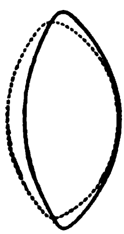
The pupils in the eyes of animals have different forms according to their habits; in the eyes of those which graze, the pupil is elongated horizontally, and in the eyes of beasts and birds of prey, it is elongated vertically.

The eyes of animals are adapted, in respect to their refractive power, to the medium which surrounds them. Animals which inhabit the water have eyes which refract much more than those of land animals. The human eye being fitted for seeing in air, is unfit for distinct vision in water, since its refractive power is nearly the same as that of water, and therefore a pencil of parallel rays from water entering the eye would scarcely be converged at all. The effect is the same as if the cornea were deprived of all its convexity.

460. Accommodation to Diminished Distance.—It has been shown (Art. 385), that as an object approaches a lens, its image moves away, and the reverse. Therefore in the eye there must be some change in order to prevent this, and keep the image distinct on the retina while the object varies its distance. In a state of rest, the eye converges to the retina only the pencils of *parallel* rays, that is, those which come from objects at great distances. Rays from near objects diverge so much that, while the eye is at rest, it cannot sufficiently converge them so that they will meet on the retina; but each conical pencil is cut off before reaching its focus, and all the points of the object are represented by overlapping circles, causing an indistinct image. The change in the eye, which fits it for seeing near objects distinctly, is called *accommodation*. This is effected by increasing the convexity of the crystalline lens, principally the front surface. The *ciliary muscle*, *m*, *m*, surrounds the crystalline, and is attached to the sclerotic coat just on the circle where it changes into the cornea. This muscle is connected with the edge of the crystalline by the circular ligament which surrounds the latter and holds it in place. When the muscle contracts, it relaxes the ligament, and the crystalline, by its own elastic force, begins to assume a more convex form, as represented by the dotted line. The eye is then accommodated for the vision of objects more or less near, according to the degree of change in the lens. On the other hand, when the ciliary muscle relaxes, the ligament again draws upon the lens to flatten it, and adapt it for the view of distant objects. In Fig. 281 these two conditions of the crystalline are more distinctly

shown. The dotted line exhibits the shape of the lens when accommodated for seeing near objects. Accompanying this action of the ciliary muscle is that of the iris, which diminishes the pupil for near objects, so as to exclude the outer and more divergent rays. The dotted lines in front of the iris represent its situation when pushed forward by the crystalline accommodated for near objects.

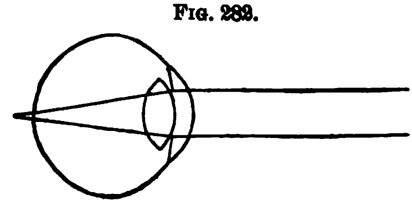
Fig. 281.



461. Long-Sightedness.—As life advances, the crystalline becomes harder and less elastic. It therefore assumes a less convex form when the ligament is relaxed, and cannot be accommodated to so short distances as in earlier years; and at length it remains so flattened in shape that only very distant objects can be seen distinctly. The eye is then said to be long-sighted, and requires a convex lens to be placed before it, to compensate for insufficient convexity in the crystalline.

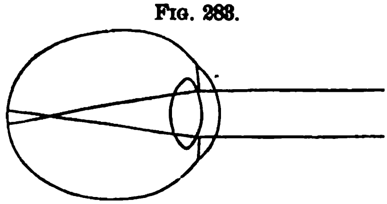
There are, however, cases of long-sightedness in early life. Such instances are found to be the result of an oblate form of the eyeball, as shown in Fig. 282; it is too short from front to back to furnish room for the convergency of the pencils, and they are cut off by the retina before reaching their focal points. In order to bring the distinct image forward upon the retina, convex glasses are needed in such cases, just as for the eyes of most people when advanced in life. As the term long-sightedness is now applied to this abnormal condition of the eye, the effect of age upon the sight is more properly called old-sightedness.

Fig. 282.



462. Short-Sightedness.—The eyes of the short-sighted have a form the reverse of that just described; the eyeball is elongated from cornea to retina (Fig. 283), resembling a prolate spheroid, so that rays parallel, or nearly so, are converged to a point before reaching the retina, and after crossing, fall on it in a circle; and the image, made up of overlapping circles instead of points, is indistinct. If this

Fig. 283.



elongation of the eyeball is extreme, an object must be brought very near, in order that its image may move back to the retina, and distinct vision be produced. This inconvenience is remedied by the use of concave lenses, which increase the divergency of the rays before they enter the eye, and thus throw their focal points further back.

In the normal condition of the eyes in early life, the nearest limit of distinct vision is about *five* inches. This limit slowly increases with advance of life, but much more slowly in some cases than others, till it is at an indefinitely great distance. The near limit of distinct vision for the short-sighted varies from *five* down to *two* inches, according to the degree of elongation in the eyeball.

463. Why an Object is Seen Erect and Single.—The image on the retina is *inverted;* and that is the very reason why the object is seen erect; the image is not the thing *seen*, but *that by means of which we see.* The impression produced at any point on the retina is referred outward in a straight line through a point near the centre of the lens, to something external as its cause; and therefore that is judged to be highest without us which makes its image lowest on the retina, and the reverse.

An object appears as *one*, though we see it by means of *two* images; but this is only one of many instances in which we have learned by experience to refer two or more sensations to one thing as the cause. Provided the images fall on parts of the retina, which in our ordinary vision *correspond* with each other, then by experience we refer both impressions to one object; but if we press one eye aside, the image falls in a new place in relation to the other, and the object seems double.

464. Indirect Vision.—The Blind Point.—To obtain a clear and satisfactory view of an object, the axes of both eyes are turned directly upon it, in which case each image is at the centre of the retina. But when the light from an object is exceedingly faint, it is better seen by *indirect vision,* that is, by looking to a point a little on one side, and especially by changing the direction of the eyes from moment to moment, so that the image may fall in various places *near* the centre of the retina. Many heavenly bodies are plainly discerned by indirect vision, which are too faint to be seen by direct vision.

In the description of the eye it was stated that the retina, as well as the choroid and the sclerotic, is perforated to allow the optic nerve to pass through. At that place there is no vision, and it is called the *blind point.* In each eye it is situated about 15° from the centre of the retina toward the other eye. Let a person

close his right eye, and with the left look at a small but conspicuous object, and then slowly turn the eye away from it toward the right; presently the object will entirely disappear, and as he looks still further to the right, it will after a moment reappear, and continue in sight till the axis of the eye is turned 70° or 80° from it. The same experiment may be tried with the right eye in the opposite direction. The reason why people do not generally notice the fact till it is pointed out, is that an object cannot disappear to both eyes at once, nor to either eye alone, when directed to the object.

465. Continuance of Impressions.—The impression which a visible object makes upon the retina continues about one-eighth or one-ninth of a second; so that if the object is removed for that length of time, and then occupies its place again, the vision is uninterrupted. A coal of fire whirled round a centre at the rate of eight or nine times per second, appears in all parts of the circumference at once. When riding in the cars, one sometimes gets a faint but apparently an uninterrupted view of the landscape beyond a board fence, by means of successive glimpses seen through the cracks between the upright boards. Two pictures, on opposite sides of a disk, are brought into view together, as parts of one and the same picture, by whirling the disk rapidly on one of its diameters. Such an instrument is called a *thaumatrope*. The *phantasmascope* is constructed on the same principle. Several pictures are painted in the sectors of a circular disk, representing the same object in a series of positions. These are viewed in a mirror through holes in the disk, as it revolves quickly in its own plane. Each glimpse which is caught whenever a hole comes before the eye, presents the object in a new attitude; and all these views are in such rapid succession that they appear like one object going through the series of movements.

466. Subjective Colors.—There are impressions on the retina of another kind, which are produced by intense lights; they continue longer, and are in respect to color unlike the objects which cause them. They are called *subjective colors*. If a particular part of the retina is for some time affected by the image of a bright colored object, and then the eyes are shut, or turned upon a white surface, the *form* appears to remain, but the *color* is complementary to that of the object; and its continuance is for a few seconds or several minutes, according to the vividness of the impression.

That portion of the retina upon which the bright colored image was formed loses sensitiveness to that particular color after a short time, and when white light falls upon the retina that particular spot is affected by the complementary color only.

467. Irradiation.—When small bodies are intensely illuminated the retina is affected somewhat beyond their proper images upon it, and the bodies consequently appear larger than they would if less bright. A white circle upon a black ground looks larger than an equal black circle upon a white ground. At new moon, when both the bright and the dark portions are visible, the crescent seems to be a part of a larger sphere than that which it accompanies.

This enlargement of the image is called irradiation.

468. Estimate of the Distance of Bodies.—

1. If objects are near, we judge of relative distance by the *inclination of the optic axes* to each other. The greater that inclination is, or, which is the same thing, the greater the change of direction in an object, as it is viewed by one eye and then by the other, the nearer it is. If objects are *very* near, we can with one eye alone judge of their distance by the degree of effort required to accommodate the eye to that distance.

2. If objects are known, we estimate their distance by the *visual angle* which they fill, having by experience learned to associate together their distance and their *apparent*, that is, their *angular* size.

3. Our judgment of distant objects is influenced by their *clearness* or *obscurity*. Mountains, and other features of a landscape, if seen for the first time when the air is remarkably pure, are estimated by us nearer than they really are; and the reverse, if the air is unusually hazy.

4. Our estimate of distance is more correct when *many objects intervene*. Hence it is that we are able to place that part of the sky which is near the horizon further from us than that which is over our heads. The apparent sky is not a hemisphere, but a flattened semi-ellipsoid.

469. Magnitude and Distance Associated.—Our judgments of distance and of magnitude are closely associated. If objects are known, we estimate their distance by their visual angle, as has been stated; but if unknown, we must first acquire our notion of their distance by some other means, and then their visual angle gives us a definite impression as to their size. And if our judgment of distance is erroneous, a corresponding error attaches to our estimate of their magnitude. An insect crawling slowly on the window, if by mistake it is supposed to be some rods beyond the window, will appear like a bird flying in the air. The moon near the horizon seems larger than above us, because we are able to locate it at a greater distance.

The apparent linear dimensions of objects are directly proportional to their actual dimensions and inversely proportional to their distances from the observer.

470. Binocular Vision.—The Stereoscope.—If objects are placed quite near us, we obtain simultaneously *two views*, which are essentially different from each other—one with one eye, and one with the other. By the right eye more of the right side, and less of the left side, is seen, than by the left eye. Also, objects in the foreground fall further to the left compared with distant objects, when seen with the right eye than when seen with the left. And we associate with these combined views the form and extent of a body, or group of bodies, particularly in respect to distance of parts from us. It is, then, by means of *vision with two eyes*, or *binocular vision*, that we are enabled to get accurate perceptions of prominence or depression of surface, reckoned in the visual direction. A picture offers no such advantage, since all its parts are on one surface, at a common distance from the eyes. But, if two perspective views of an object should be prepared, differing as those views do, which are seen by the two eyes, and if the right eye could then see only the right-hand view, and the left eye only the left-hand view, and if, furthermore, these two views could be made to appear on one and the same ground, the vision would then be the same as is obtained of the real object by both eyes. This is effected by the *stereoscope*. *Two* photographic views are taken, in directions which make a small angle with each other, and these views are seen at once by the two eyes respectively, through a pair of half-lenses, placed with their thin edges toward each other, so as to turn the visual pencils away from each other, as though they emanated from one object. An appearance of *relief* and *reality* is thus given to superficial pictures, precisely like that obtained from viewing the objects themselves.

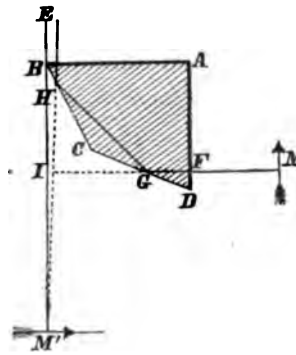
CHAPTER IX

OPTICAL INSTRUMENTS.

471. The Camera Lucida.—This is a four-sided prism, so contrived as to form an apparent image at a surface on which that image may be copied, the surface and image being both visible at the same time. It has the form represented by the section in

Fig. 284; $A = 90°$, $C = 135°$; B and D, of any convenient size, their sum of course $= 135°$. A pencil of light from the object M, falling perpendicularly on $A\ D$, proceeds on, and makes, with $D\ C$, an angle equal to the complement of D. After suffering total reflection at G, and again at H, its direction $H\ E$ is perpendicular to $M\ F$. For, produce $M\ F$, and $E'H$, till they intersect in I; then, since $C = 135°$, $C\ G\ H + C\ H\ G = 45°$; but $I\ G\ H = 2\ C\ G\ H$, and $I\ H\ G = 2\ C\ H\ G$; $\therefore I\ G\ H + I\ H\ G = 90°$; $\therefore I = 90°$. Therefore $H\ E$ emerges at right angles to $A\ B$, and is not refracted. Now, if the pupil of the eye be brought over the edge B, so that, while $E\ H$ enters, there may also enter a pencil from the surface at M', then both the surface M' and the object M will be seen coinciding with each other, and the hand may therefore sketch M on the surface at M'. The reason for two reflections of the light is, that the inversion produced by one reflection may be restored by the second.

FIG. 284.



One of the most useful applications of the camera lucida is in connection with the compound microscope, where it is employed in copying with exactness the forms of natural objects, too small to be at all visible to the naked eye.

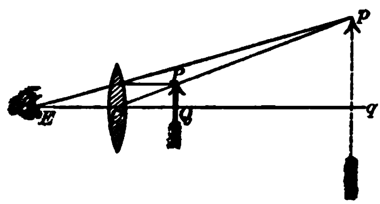
472. The Microscope.—This is an instrument for *viewing minute* objects. The nearer an object is brought to the eye, the larger is the angle which it fills, and therefore the more perfect is the view, provided the rays of each pencil are converged to a point on the retina. But if the object is nearer than the limit of distinct vision, the eye is unable to produce sufficient convergency. If the letters of a book are brought close to the eye, they become blurred and wholly illegible. But let a pin-hole be pricked through paper, and interposed between the eye and the letters, and, though faint, they are *distinct* and *much enlarged.* The *distinctness* is owing to the fact that the outer rays, which are most divergent, are excluded, and the eye is able to converge the few central rays of each pencil to a focus. The letters appear *magnified,* because they are so near, and fill a large angle. The microscope utilizes these excluded rays, and renders the image not only large and distinct, but luminous.

473. The Single Microscope.—The single microscope is merely a convex lens. It aids the eye in converging the rays,

which come from an object placed between the lens and its focus, so that a distinct and luminous image may be formed on the retina.

Let $P\ Q$ (Fig. 285) be the object, and $p\ q$ its image. The image is at the *distance of distinct vision* (24 cm.), and subtends a larger angle than the object would at the same distance; it therefore appears larger. Since the eye is always placed on the lens, the magnifying power is $\frac{p\ q}{P\ Q} = \frac{C\ q}{C\ Q}$. Let F = focal length of the lens, $C\ Q = p$, and $C\ q = -q$ (negative because the image is virtual). Art. 383 gives us

Fig. 285.



$$\frac{1}{F} = \frac{1}{p} - \frac{1}{q},$$

whence $\frac{q}{p} = \frac{q}{F} + 1 = \frac{C\,q}{C\ Q} = \frac{p\ q}{P\ Q}$ = magnifying power. Since q = 24 cm., the magnifying power is greater the smaller the focal length of the lens. *E.g.*, if F = 3 cm. the magnification is 9.

Though the focal distance of a lens may be made as small as we please, yet a practical limit to the magnifying power is very soon reached.

1. The *field of view*, that is, the extent of surface which can be seen at once, diminishes as the power is increased.

2. Spherical aberration increases rapidly, because the outer rays are very divergent. Hence the necessity of diminishing the aperture of the lens, in order to exclude the most divergent rays.

3. It is more difficult to illuminate the object as the focal length of the lens becomes less; and this difficulty becomes a greater evil on account of the necessity of diminishing the aperture in order to reduce the spherical aberration.

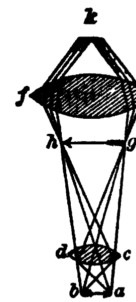
To lessen the spherical aberration, two or more plano-convex lenses are used, close together. When this is done, the plane face of each lens is turned towards the object. Although each lens is less convex than the simple lens which together they replace, yet their joint magnifying power is as great, and with a less amount of spherical aberration, since the first lens draws towards the axis the rays which fall on the second lens. This combination of lenses is known as *Wollaston's doublet*.

Magnifying-glasses are single microscopes of low power, such as

are used by watchmakers. Lenses of still lower power and several inches in diameter are used for viewing pictures.

474. The Compound Microscope.—It is so called because it consists of two parts, an object-glass, by which a real and magnified image is formed, and an eye-glass, by which that image is again magnified. Its general principle may be explained by Fig. 286, in which *a b* is the small object, *c d* the object-glass, and *e f* the eye-glass. Let *a b* be a little beyond the principal focus of *c d*, and then the image *g h* will be real, on the opposite side of *c d*, and larger than *a b*. Now apply *e f* as a single microscope for viewing *g h*, as though it were an object of comparatively large size. Let *g h* be at the principal focus of *e f*, so that the rays of each pencil shall be parallel; they will, therefore, come to the eye at *k*, from an *apparent* image on the same side as the *real* one, *g h*; and the extreme pencils, *e k*, *f k*, if produced backward, will include the image between them, *e k f* being the angle which it fills.

FIG. 286.



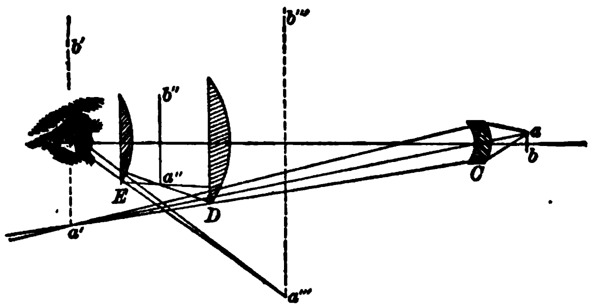
475. The Magnifying Power.—The *magnifying power* of the compound microscope is estimated by compounding two ratios; first, the distance of the image from the object-glass, to the distance of the object from the same; and secondly, the limit of distinct vision to the distance of the image from the eye-glass. For the image itself is enlarged in the first ratio (Art. 385); and the eye-glass enlarges that image in the second ratio (Art. 473). The advantage of this form over the single microscope is not so much that a great magnifying power is obtained, as that a given magnifying power is accompanied by a larger field of view.

476. Modern Improvements.—Great improvements have been made in the compound microscope, principally by combining lenses in such a manner as greatly to reduce the chromatic and spherical aberrations. The object-glass generally consists of one, two, or three achromatic pairs of lenses. The eye-piece usually contains two plano-convex lenses, a combination which is found to be the most favorable for diminishing the spherical aberration, and for enlarging the field of view.

In Fig. 287 let *a b* be the object, *C* an achromatic lens, called the *objective*, *D* the field lens (so called because it enlarges the field of view by bending the pencil which would come to a focus at *a′*, and pass below the eye lens, so that it may come to a focus at *a″*, and thence pass into the eye lens), *E* the eye lens which

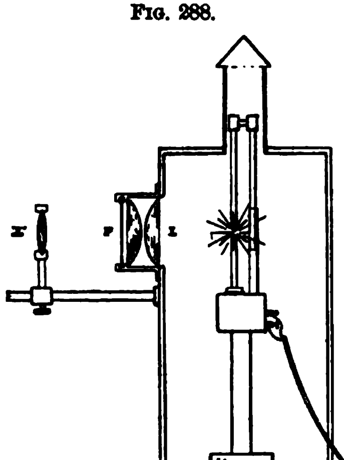
renders the rays of the pencil so nearly parallel that the eye receives them as though coming from the point a'''.

FIG. 287.



The method of determining the magnifying power given in the last paragraph, is not applicable in this form of instrument, but an experimental determination is made as follows: A very finely-divided scale is placed under the microscope; a mirror, from which a small part of the silvering has been removed, is placed near the eye-piece at an angle of 45° with the axis of the instrument, and behind this at the distance of ordinary distinct vision, about ten inches, and visible through the unsilvered part of the mirror, is placed a second scale like the first; the number of divisions of the second scale covered by one division seen through the microscope by reflection gives the power. Any change in the relative positions of the lenses, changes the magnifying power.

FIG. 288.



477. The Projecting Lantern.—Projecting lanterns, stereopticons, and magic lanterns are all constructed upon the same plan. The principle involved is shown in Fig. 288. An electric arc-lamp, lime or other powerful light is placed in a box. This box is provided with holes which allow ventilation but prevent the escape of light. In one side of the box, at the same height as the arc, is an opening in which is placed a *condenser*, *L*. The condenser consists of two plano-convex lenses with their convex surfaces opposed to each other. (A double convex lens is sometimes used in place of the combina-

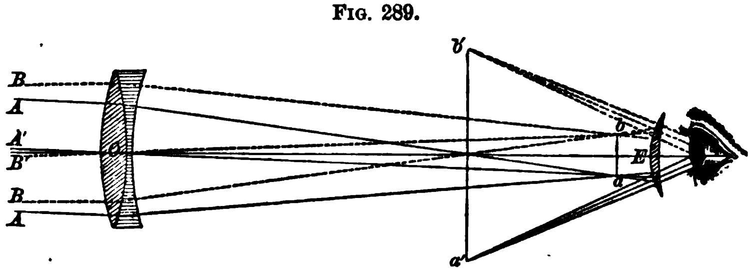
tion.) These lenses are of short focal length and serve to concentrate the rays from the lamp upon the slide P. Upon the slide is photographed the picture or object to be projected. After passing the slide, the rays are brought to a focus on a distant screen by the *projecting lens* L'. This lens is so adjusted that the slide P and the screen, upon which it is projected, are at conjugate foci. This is, therefore, a little outside the focal length of L'. It can be readily seen that the diameter of the image on the screen will bear the same ratio to the diameter of the slide as their respective distances from the lens L'.

478. The Telescope.—The telescope aids in *viewing distant* bodies. An image of the distant body is first formed in the principal focus of a convex lens or a concave mirror; and then a microscope is employed to magnify that image as though it were a small body. The image is much more luminous than that formed in the eye, when looking at the heavenly body, because there is concentrated in the former the large beam of light which falls upon the lens or mirror, while the latter is formed by the slender pencil only which enters the pupil of the eye. If the image in a telescope is formed by a lens, the instrument is called a *refracting telescope;* but if by a mirror, a *reflecting telescope.*

479. The Astronomical Telescope.—This is the most simple of the refracting telescopes, consisting of a lens to form an image of the heavenly body, and a single microscope for magnifying that image. The former is called the *object-glass,* the latter the *eye-glass.*

Let a (Fig. 289) be the image of some point of a heavenly body, the divergent rays from which, marked A A' A, are practi-

FIG. 289.



cally parallel, and b the image of the point B B' B. As the rays forming these images are parallel rays, a b is at the principal focus of the object-glass O. The eye lens E receives the diver-

gent pencils from a and b, bends them so that they enter the eye as parallel or nearly parallel beams coming apparently from the direction of a' and b'. The image $a\ b$ is situated at the principal focus of E, the distance between the lenses O and E being the sum of their principal focal distances.

480. The Powers of the Telescope.—The *magnifying power* of the astronomical telescope is expressed by the ratio of the *focal distance of the object-glass to that of the eye-glass.* For (Fig. 289) the object as it would be seen by the eye if placed at O fills the angle $A'\ O\ B'$ between the axes of its extreme pencils. But, since the axes cross each other in straight lines at the optic centre of the lens, $A'\ O\ B' = a\ O\ b$. Therefore, to an eye placed at the object-glass, the image, $a\ b$, appears just as large as the object; while at the eye-glass it appears as much larger in diameter as the distance is less.

Since no simple eye lenses are used, and as the equivalent power of the compound eye-piece is not readily found, the following practical method of finding the power of an astronomical telescope is of use:

Adjust the eye-piece so that a sharp and clear view of some very distant object may be obtained; remove the object-glass, and in its place put an opaque card disk in which is cut an opening of the shape of a very flat isosceles triangle, whose base is nearly as long as the diameter of the object-glass; receive an image of this opening upon a translucent glass or paper screen held close to the eye-piece, and measure the base of the image very exactly; the length of the base of the opening divided by the base of its image is the power. If any of the rays from the opening are cut off by diaphragms in the tube the imperfection of the image will make known the difficulty, which must be removed.

The field of view is determined thus: Direct the telescope to a star on or near the celestial equator, and note the time in seconds which the star occupies in passing across the diameter of the field of view; divide this time by 4 and the quotient will be the diameter of the field in minutes of arc, because a star on the equator moves through one minute of arc in four seconds of time.

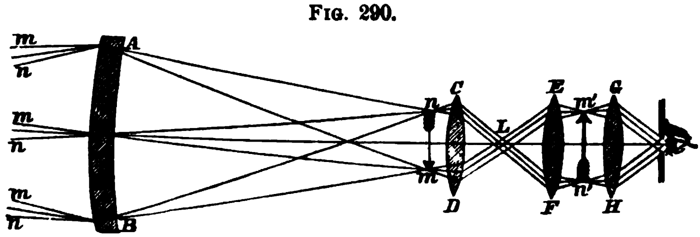
The *illuminating power* is important for objects which shed a very feeble light on account of their immense distance. This power depends on the size of the beam, that is, on the aperture of the object-glass.

The *defining power* is the power of giving a clear and sharply-defined image, without which both the other powers are useless. And it is the power of producing a well-defined image which limits both of the other powers. For every attempt to increase

the magnifying power by giving a large ratio to the focal lengths of the object-glass and the eye-glass, or to increase the illuminating power by enlarging the object-glass, increases the difficulties in the way of getting a perfect image. These difficulties are three —the spherical aberration (Art. 388), the chromatic aberration (Art. 402), and unequal densities in the glass. The third difficulty is a very serious one, especially in large lenses.

481. The Terrestrial Telescope.—In order to secure simplicity, and thus the highest excellence, in the astronomical telescope, the image is allowed to be *inverted*, which circumstance is of no importance in viewing heavenly bodies. But, for terrestrial objects, it would be a serious inconvenience; and, therefore, a *terrestrial telescope*, or *spy-glass*, has additional lenses for the purpose of forming a second image, inverted, compared with the first, and, therefore, erect, compared with the object. In Fig. 290, *m m m* represent a pencil of rays from the top of a distant object, and

FIG. 290.



n n n from the bottom; *A B*, the object-glass; *m n*, the first image; *C D*, the first eye-glass, which converges the pencils of parallel rays to *L*. Instead of placing the eye at *L*, the pencils are allowed to cross and fall on the second eye-glass, *E F*, by which the rays of each pencil are converged to a point in the second image, *m′ n′*, which is viewed by the third eye-glass, *G H*. The second and third lenses are commonly of equal focal length, and add nothing to the magnifying power.

Such instruments are usually of a portable size, and hence the aberrations are corrected with comparative ease, by the methods already described. The spy-glass, for convenient transportation, is made of a series of tubes, which slide together in a very compact form.

If the lenses *C D* and *E F* are of the same power they do not affect the power of the telescope, which may then be represented as in the astronomical telescope by $\frac{F}{f}$.

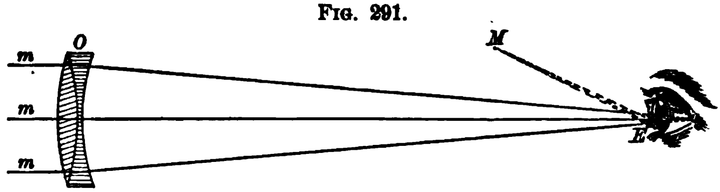
To determine the power practically, look at some distant scale of equal parts, a brick wall for instance, and keeping both eyes

open, note how many bricks as seen by the unaided eye are covered by the image of one brick as seen through the telescope; this number so covered is the expression for the power. If one space, for instance, seen through the telescope, covers twenty spaces seen with the unaided eye, the telescope magnifies twenty diameters.

482. Galileo's Telescope.—This was the first form of telescope, having been invented by Galileo, whose name it therefore bears. It differs from the common astronomical telescope in having for the eye-glass a *concave* instead of a convex lens, which receives the rays at such a distance from the focus to which they tend, as to render them parallel.

Thus the rays *m m m* (Fig. 291), from a point at the top of the object, are converged by the object-glass *O* towards a focus *a*;

FIG. 291.



but before meeting at *a* they fall upon a concave eye-lens *E*, and are rendered parallel or slightly divergent, as though they came from a point in the direction indicated by *M*. The point from which the rays *m m m* proceeded, and its virtual image *M*, are both on the same side of the axis of the instrument, and there is no inversion.

It is obvious that, since the pencils diverge, only the central ones, within the size of the pupil, can enter the eye. This circumstance exceedingly limits the field of view, and unfits the instrument for telescopic use. It is employed for opera-glasses, having a power usually of only *two* or *three* diameters.

The expression for the power is $\frac{F}{f}$, as in the preceding forms. The power may be very readily determined practically as in the case of the terrestrial telescope.

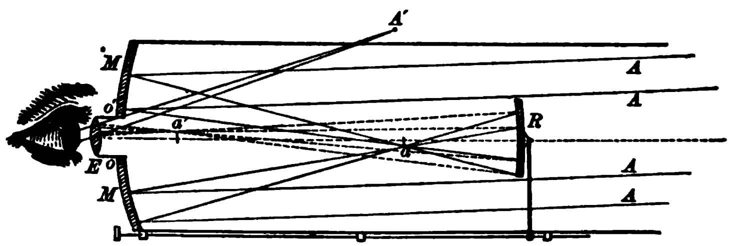
483. The Gregorian Telescope.—This is the most frequent form of reflecting telescope, and receives its name from the inventor, Dr. Gregory, of Scotland.

Let *A* (Fig 292) be a point of a very distant body from which rays, practically parallel, fall upon the large concave mirror *M*, which is perforated through the middle *o o'*; these are converged to the principal focus *a*, and passing this point, diverging again, are received by the small concave reflector *R* of short focus, and

are made to converge to a', forming a real image; thence the rays diverge once more, and falling upon the eye-glass E, are refracted as though they came from an object in the direction A'.

The *Cassegrainian* telescope differs from the Gregorian in having a *convex* reflector in place of the *concave* R; this is so

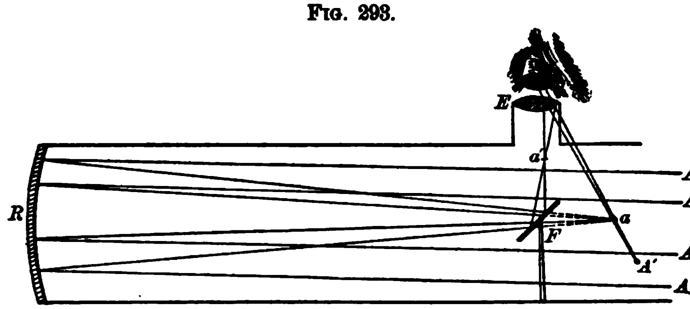
FIG. 292.



placed as to receive the rays before they reach the focus a, and, by rendering them less convergent, bring them to a focus at the place of the image a', but upon the same side of the axis as a.

484. The Newtonian Telescope.—R is a concave reflector (Fig. 293), F a plane mirror called a *flat*, E the convex eye-glass. Rays from some point A of a distant object are converged by R towards the principal focus a; they are intercepted by the *flat* F and turned aside, without change of convergency, to the

FIG. 293.



focus a', passing which they fall upon the eye-lens E, and enter the eye as though they came from the direction A'. The magnifying power =

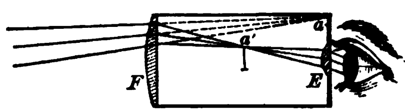
$$\frac{\text{focal length of reflector}}{\text{focal length of eye-glass}}.$$

485. The Herschelian Telescope.—Sir William Herschel modified the Newtonian by dispensing with the small reflector F, and inclining the large speculum R, so as to form the image near the edge of the tube, where the eye-glass is attached. Thus, the

observer is situated with his back to the object. The speculum of Herschel's telescope was about four feet in diameter, and weighed more than 2,000 pounds, and its focal length was forty feet. The Earl of Rosse has since constructed a Herschelian telescope having an aperture of *six* feet, and a focal length of *fifty* feet. The magnifying power is the same as in the Newtonian.

486. Eye-pieces, or Oculars.—The negative, or Huyghenian, eye-piece consists of two plano-convex lenses of crown glass, F and E (Fig. 294), the convex surfaces being turned toward the object-glass. A pencil of rays from the object-glass, converging to a principal focus a, is bent from its course by F and brought to a focus a', halfway between the two lenses. The image formed at a' is then viewed by the eye-lens E as usual.

FIG. 294.

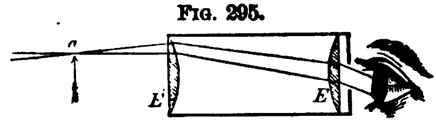


This eye-piece is called negative because it is adapted to rays already converging. The focal length of F is three times that of E, and the distance between the lenses is one-half the sum of the focal lengths.

This combination is also achromatic. In practice the object-glass, used in connection with the Huyghens' eye-piece, is an achromatic lens, which, however, is *over-corrected*. The rays after passing it are refracted by F so as to form a series of colored images around a'. Owing to the over-correction the red image is formed at the left of a' and the violet at the right. These images are then made to coalesce by the refraction of the lens E. Huyghens was not aware of this peculiarity of his eye-piece.

The *positive*, or Ramsden, eye-piece consists of two plano-convex lenses E' and E (Fig. 295) with the convex surfaces turned toward each other. The rays from the object-glass are focussed at a and thence pass to the eye as indicated in the figure. Two lenses are used instead of one, in order more easily and perfectly to correct spherical aberration. E' and E are of equal focal lengths, and the distance between them is two-thirds the focal length of one of them. This combination is not achromatic. It is always used when spider lines are placed in the focus of the object-glass for purposes of exact measurement.

FIG. 295.



PART VI.

HEAT.

CHAPTER I.

EXPANSION BY HEAT.—THE THERMOMETER.

487. Nature of Heat.—There is abundant reason for believing that heat consists of exceedingly minute and rapid vibrations of ordinary matter and of the ether which fills all space. It is to be regarded as one of the modes of *motion*, which may be caused by any kind of force, and which may be made a measure of that force. Heat affects only one of our senses, that of feeling. Its increase produces the sensation of warmth, and its diminution that of cold.

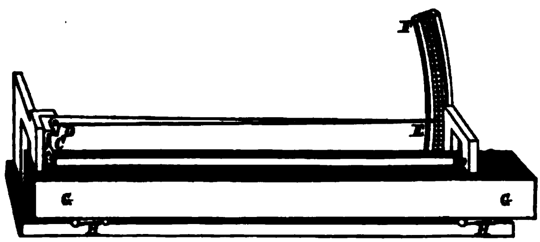
488. Expansion and Contraction by Heat and Cold.—It is found to be a fact almost without exception, that as bodies are heated they are expanded, and that they contract as they are cooled. It is easy to conceive that the vibratory motion of the several molecules of a body compels them to recede from each other, and to recede the more as the vibration becomes more violent. Although the change in magnitude is generally very small, yet it is rendered visible by special contrivances, and is made the means of measuring temperature.

489. Expansion of Solids.—When the expansion of a solid is considered simply in one dimension, it is called *linear* expansion; in two dimensions only, *superficial* expansion; in all three dimensions, *cubical* expansion.

The linear expansion of a metallic rod is readily made visible by an instrument called the *pyrometer*, which magnifies the motion. The end *A* of the rod *A B* (Fig. 296) is held in place by a screw. The end *B* rests against the short arm of the lever *C*, the longer arm of which bears on the arm *D* of the long bent lever *D E*; this serves as an index to the graduated arc *E F*. The long metallic dish *G G*, being raised on the hinges *H H*, so as to

enclose the bar $A\ B$, and then filled with hot water, the bar instantly expands, and raises the index along the arc $E\ F$.

FIG. 296.



490. Coefficient of Expansion.—The *coefficient* of linear expansion of a given substance is the fractional increase of its length, when its temperature is raised *one degree*. But since this increase is generally somewhat greater at higher temperatures, the coefficients of expansion given in tables usually refer to a temperature at or near the freezing point of water. Thus the coefficient of expansion for silver is 0.000019097; by which is meant that a silver bar one foot long at 0° C. becomes 1.000019097 ft. in length at 1° C.

The coefficient of superficial expansion is *twice*, and that of cubical expansion *three times* as great as the coefficient of linear expansion. For, suppose c to be the coefficient of linear expansion; then if the edge of a cube is 1, and the temperature is raised 1°, the edge becomes $1 + c$, and the area of one side becomes $(1 + c)^2 = 1 + 2c + c^2$, and the volume $(1+c)^3 = 1+3c+3c^2+c^3$. But as c is very small, the higher powers may be neglected, and the area is $1+2c$, and the volume is $1+3c$; that is, the coefficient of superficial expansion is $2c$, and that of cubical expansion is $3c$, as stated above.

491. The Coefficient of Expansion differs in different Substances.—Copper expands nearly twice as much as platinum for a given increase of temperature; the ratio of expansion in steel and brass is about as 61 to 100. This ratio is employed in the construction of the compensation pendulum (Art. 164).

If two thin slips of metal of different expansibility be soldered together so as to make a slip of double thickness, it will bend one way and the other by changes of temperature. If it is straight at a certain temperature, heating will bend it so as to bring the most expansible metal on the convex side; and cooling will bend it in the opposite direction; and the degree of flexure will be according to the degree of change in temperature. Compensation in clocks and watches is sometimes effected on this plan. If the

compound slip has the form of a helix, with the most expansible metal on the inside, heating will begin to uncoil it, and cooling, to coil it closer. A very sensitive thermometer, known as Breguet's thermometer, is constructed on this principle.

As notable exceptions to the general rule that solids expand when heated, may be mentioned stretched India-rubber, and also Rose's fusible metal, an alloy of 2 parts bismuth, 1 part lead, and 1 part tin; the latter compound expands up to 44° C., then rapidly contracts up to 69° C., which is the temperature of maximum density, and again expands till it melts at 94° C.

492. The Strength of the Thermal Force.—It is found that the force exerted by a body, when expanding by heat or contracting by cold, is equal to the mechanical force necessary to expand or compress the body to the same degree. The force is therefore very great. If the rails were to be fitted tightly end to end on a railroad, they would be forced out of their places by expansion in warm weather, and the track ruined. The tire of a carriage wheel is heated till it is too large, and then put upon the wheel; when cool, it draws together the several parts with great firmness. In repeated instances, the walls of a building, when they have begun to spread by the lateral pressure of an arched roof, have been drawn together by the force of contraction in cooling. A series of iron rods being passed across the building through the upper part of the walls, and broad nuts being screwed upon the ends, the alternate bars are expanded by the heat of lamps, and the nuts tightened. Then, when they cool, they draw the walls toward each other. The remaining bars are then treated in the same manner, and the process is repeated till the walls are restored to their vertical position and secured.

493. Expansion of Liquids.—As liquids have no permanent form, the coefficient of expansion for them is always understood to be that of cubical expansion. There is a practical difficulty in the way of finding the coefficient for liquids, because they must be enclosed in some solid, which also expands by heat. Hence, the *apparent* expansion must be corrected by allowing for the expansion of the inclosing solid, before the coefficient of *absolute* expansion is known.

This fact is illustrated by the following experiment. Fill the bulb and part of the stem of a large thermometer tube with a colored liquid, and then plunge the bulb quickly into hot water; the first effect is, that the liquid *falls*, as if it were cooled; after a moment it begins to rise, and continues to do so till it attains the temperature of the hot water. The first movement is caused by the expansion of the glass, which is heated so as to enlarge its

capacity and let down the liquid before the heat has penetrated the latter. It is obvious that what is rendered visible in this case, must always be true when a liquid is heated—namely, that the vessel itself is enlarged, and therefore that the rise of the liquid shows only the difference of the two expansions. Ingenious methods have been devised for obtaining the coefficients of absolute expansion of liquids, and the results are to be found in tables on this subject.

From the examination of such tables we learn: (1) That liquids expand more than solids for a given increase of temperature; (2) that the coefficient of expansion increases with the rise of temperature; (3) that the more volatile the liquid the more rapidly will it expand for a given rise of temperature.

494. Exceptional Case.—There is a very important exception to the general law of expansion by heat and contraction by cold, in the case of water just above the freezing point. If water be cooled down from its boiling point, it continually contracts till it reaches 39.1° F. or 3.94° C., when it begins to expand, and continues to expand till it freezes at 32° F. or 0° C. On the other hand, if water at 32° F. be heated, it contracts till it reaches 39.1° F. or 3.94° C., when it commences to expand. Therefore the density of water is greatest at the point where this change occurs. Different experimenters vary a little as to its exact place, but it is usually called 4° C., or 39° F.

The importance of this exception is seen in the fact that ice forms on the *surface* of water, and continues to float until it is again melted. As the cold of winter comes on, the upper stratum of a lake grows more dense and sinks; and this process continues till the temperature of the surface reaches 39° F., when it is arrested. Below that temperature the surface grows lighter as it becomes colder, till ice is formed, which shields the water beneath from the severe cold of the air above.

As in solids so in liquids, the thermal force is very great. Suppose mercury to be expanded by raising its temperature one degree, it would require more than 300 pounds to the square inch to compress it to its former volume.

495. Expansion of Gases.—The gases expand by heat more rapidly and more regularly than solids and liquids. The large expansion and contraction of air is made visible by immersing the open end of a large thermometer tube in colored liquid. When the bulb is warmed, bubbles of air are forced out and rise to the top of the liquid; when it is cooled, the air contracts and the liquid rises rapidly in the tube.

Gases, *at a constant pressure*, expand much more than liquids

or solids for a given increment of temperature. All gases, at temperatures much above that of liquefaction, have almost exactly the same coefficients of expansion. The coefficient of expansion for air is $\frac{1}{273}$ from 0° C. to 1° C. This coefficient increases *slightly* with increase of temperature and pressure.

To find the volume of any gas at 0° C., let v be the known volume at $t°$ C., also let v' be the required volume at 0° C., then

$$v = v'\left(1 + \tfrac{1}{273}\,t\right),$$

from which we have $v' = \dfrac{v}{1 + \frac{1}{273}\,t}.$

If there is a change of pressure, then, since the tensions or pressures are inversely as the volumes, the temperatures being the same (Art. 237), we have

$$v'' : v' :: p' : p,$$

in which v'' is the volume at 0° C. and barometric pressure of 760 mm., v' the volume at pressure p', and p the normal pressure 760 mm.; from which we get, by substituting for v' its value above,

$$v'' = \frac{v}{1 + \frac{1}{273}\,t} \times \frac{p'}{p}.$$

496. The Thermometer.—This instrument measures the degree of heat, or the *temperature*, of the medium around it, by the expansion and contraction of some substance. The substance commonly employed is mercury. The liquid, being inclosed in a glass bulb, can expand only by rising in the fine bore of the stem, where very small changes of volume are rendered visible. A scale is attached to the stem for reading the degrees of temperature.

The graduation of the thermometer must begin with the fixing of two important points by natural phenomena, the melting of ice and boiling of water. When the bulb is plunged into powdered ice, the point at which the column settles is the *freezing-point* of the thermometer. And if it is placed in steam under the mean atmospheric pressure, the mercury indicates the *boiling-point*. Between these two points, namely 0° and 100° C., there must be 100°, and the scale is graduated accordingly. As the bore of the tube is not likely to be exactly equal in all parts, the length of the degrees should vary inversely as the area of the cross-section. The necessary correction is determined by moving a short column of mercury along the different parts and comparing the lengths occupied by it. The degrees in the several parts must vary in the ratio of these lengths.

The zero of the scale tends to rise for some time after the thermometer is made, the change amounting to more than 2° in

some instances, and therefore the instrument should not be used for at least six months after construction. The zero may also be displaced by subjecting the instrument to high temperatures.

497. Different Systems of Graduation.—There are in use three kinds of thermometer scale, Fahrenheit's, Reaumur's, and the Centigrade or Celsius. In Fahrenheit's, the freezing point of water is called 32°, and the boiling point, 212°; in Reaumur's, the freezing point is called 0°, and the boiling point 80°; in the Centigrade, the freezing point 0°, and the boiling point 100°. In a scientific point of view, the Centigrade is preferable to either of the others, but Fahrenheit's is generally used in this country. The letter F., R., or C., appended to a number of degrees, indicates the scale intended. In this country, F. is understood if no letter is used.

498. To Reduce from one Scale to Another.—Since the zero of Fahrenheit's scale is 32° below the freezing point, while in both of the others it is at the freezing point, 32° must always be subtracted from any temperature according to Fahrenheit, in order to find its relation to the zero of the other scales. Then, since 212° − 32° (= 180°) F. are equal to 80° R., and to 100° C., the formula for changing F. to R. is $\frac{4}{9}$ (F. − 32) = R.; and for changing F. to C., it is $\frac{5}{9}$ (F. − 32) = C. Hence, to change R. to F., we have $\frac{9}{4}$ R. + 32 = F.; and to change C to F., $\frac{9}{5}$ C. + 32 = F.

Mercury congeals at about −38.8° C.; therefore, for temperatures lower than that, alcohol is used, which does not congeal at any known temperature.

Above 100° C. the indications of the mercurial thermometer are not exact.

499. Absolute Zero of Temperature.—At a temperature of 273° C. the volume of a gas is double its volume at 0° C. (Art. 495). Suppose that instead of raising the temperature, we lower it; for a fall from 0° to −1° C. the volume contracts $\frac{1}{273}$, and for a fall of 273° it must contract $\frac{273}{273}$; that is to say, the volume would disappear entirely. That the contraction *would* go on to −273° C. is not asserted; but on the supposition that the law of contraction would hold, we fix the temperature −273° as that at which all vibrations would cease, and at which consequently there could be no heat whatever. The absolute zero more exactly given is −273.7° C., and −460.66° F. The absolute temperature is found by adding these readings, with signs changed, to the respective readings of the mercurial thermometer. As both Fahrenheit and

Centigrade thermometers are in use, both will be referred to in the text, as indicated, that the student may become familar with both systems of graduation.

CHAPTER II.

PASSAGE OF HEAT THROUGH MATTER AND SPACE.

500. Heat is Communicated in Several Ways.—

1. By *conduction*. This is the slow progress of the vibratory motion from places of higher to places of lower temperature in the same body.

2. By *convection*. This mode of communication takes place only in *fluids*. When the particles are expanded by heat, they are pressed upward by others which are colder and therefore specifically heavier. Heat is thus conveyed from place to place by the motion of the heated matter, though the ultimate transfer of heat may still take place by conduction.

3. By *radiation*. Heat is said to be *radiated* when the vibratory motion is transmitted from the source with great swiftness through the ether which fills space. Its velocity is the same as that of light. The motion is propagated in straight lines in every direction, and each line is called a *ray* of heat. We feel the rays of heat from the sun or a fire, when no object intervenes between it and ourselves.

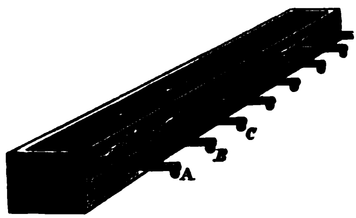
501. Conduction of Heat by Solids.—Conducted heat passes through bodies very slowly, and yet at very different rates in different bodies. Those in which heat is conducted most rapidly, are called good conductors, as the common metals; those in which it passes slowly, are called poor conductors, as glass and wood. In general, the bodies which are good conductors of heat, are also good conductors of electricity; thus calling the conductivity for electricity E, and for heat H, and using silver as the standard, we find—

Silver..........	E =	100	H =	100	Iron	E =	15	H =	16
Copper.........	"	90	"	90	Lead	"	9	"	8
Gold...........	"	59	"	53	German silver	"	8	"	8
Brass..........	"	22	"	24	Bismuth	"	1	"	1

Let rods of different metals and other substances, A, B, C,

&c. (Fig. 297), all of the same length, be inserted with water-tight joints in the side of a wooden vessel. Then attach by wax a marble under the end of each rod, and fill the vessel with boiling water. The marbles will fall by the melting of the wax, not at the same, but at different times, showing that the heat reaches some of them sooner than others. It will be seen, however, in the chapter on specific heat, that the order in which they fall is not necessarily the order of conducting power.

FIG. 297.



The amount of heat conducted through a thin lamina is directly proportional to the area, to the time during which it flows, to the difference of temperature at the two surfaces, and to the conductivity of the substance; and is inversely proportional to the thickness of the lamina.

502. Effects of Molecular Arrangement.—Organic substances usually conduct heat poorly; and bodies having a structural arrangement which differs in different directions, are not likely to conduct equally well in all directions. Thus, let two thin plates be cut from the same crystal, one, *A* (Fig. 298), per-

FIG. 298.

pendicular, and the other, *B*, parallel to the optic axis. Let a hole be drilled through the centre of each, and after a lamina of wax has been spread over the crystal, let a hot wire be inserted in it. On the plate *A*, the melting of the wax will advance in a circle, showing equal conducting power in all directions in the transverse section. In the plate *B*, it will advance in an elliptical form, the major axis being parallel to the optic axis of the crystal, proving the best conduction to be in that direction.

A block of wood cut from one side of the trunk of a tree, conducts most perfectly in the direction of the fibre, and least in a direction which is tangent to the annual rings and perpendicular to the fibre, and in an intermediate degree in the direction of the radius of the rings.

503. Conduction by Fluids.—Fluids, both liquid and gaseous, are in general very poor conductors. Water, for example, can be made to boil at the top of a vessel, while a cake of ice is fastened within it a few inches below the surface. If thermometers are placed at different depths, while the water boils at the top, there is discovered to be a very slight conduction of heat downward. The gases conduct even more imperfectly than liquids.

It will be seen hereafter (Art. 505) that a mass of fluid becomes heated by convection, not by conduction.

504. Illustrations of Difference in Conductive Power.—In a room where all articles are of equal temperature, some feel much colder than others, simply because they conduct the heat from the hand more rapidly; painted wood feels colder than woolen cloth, and marble colder still. If the temperature were higher than that of the blood, then the marble would seem the hottest, and the cloth the coolest, because of the same difference of conduction *to* the hand.

Our clothing does not impart warmth to us, but, by its non-conducting property, prevents the vital warmth from being wasted by radiation or conduction. If the air were hotter than our blood, the same clothing would serve to keep us cool.

A pitcher of water can be kept cool much longer in a hot day, if wrapped in a few thicknesses of cloth; for these prevent the heat of the air from being conducted to the water. In the same way ice may be prevented from melting rapidly.

The vibrations of heat, like those of sound, are greatly interrupted in their progress by want of continuity in the material. Any substance is rendered a much poorer conductor by being in the condition of a powder or fibre. Ashes, sand, sawdust, wool, fur, hair, &c., owe much of their non-conducting quality to the innumerable surfaces which heat must meet with in being transmitted through them.

Davy's safety lamp is a practical application of conduction. A wire gauze surrounds the lamp, and the air which supplies the flame with oxygen can only reach it by passing through the gauze. A naked flame would ignite the *fire damp* of the mines; but though the fire damp may ignite after passing the gauze and may fill the whole lamp with a body of flame, yet, owing to the cooling effected by the conduction of the wires, the gases on the outside are not raised to the temperature of ignition; thus warning, and time for escape from danger, are given.

505. Convection of Heat.—Liquids and gases are heated almost entirely by convection. As heat is applied to the sides and bottom of a vessel of water, the heated particles become

specifically lighter, and are crowded up by heavier ones which take their place. There is thus a constant circulation going on which tends to equalize the temperature of the whole, by bringing the hot portions into contact with the colder, and thus greatly facilitating the *conduction* of heat among the molecules.

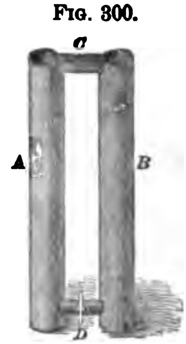
Fig. 299.

This motion is made visible in a glass vessel, by putting into the water some opaque powder of nearly the same density as water. Ascending currents are seen over the part most heated, and descending currents in the parts farthest from the heat, as represented in Fig. 299. The ocean has perpetual currents caused in a similar manner. The hottest portions flow away from the tropical toward the polar latitudes, while at greater depth the cold waters of high latitudes flow back toward the tropics.

For a like reason, the air is constantly in motion. The atmospheric currents on the earth have been considered in Chapter III. of Pneumatics.

506. Determination of the Temperature of Water at its Maximum Density.—The apparatus used by Joule in his research is represented in outline in Fig. 300, in which *A* and *B* are cylinders 4½ feet high and 6 inches in diameter; the open trough *C* connects them at top, and a large tube, with stop-cock *D*, connects them at the bottom. When the cylinders were filled so that there was a free flow through the trough *C*, any difference of density in *A* and *B* would produce a convection current through *D* and *C*, and the existence of such current in *C* was made known by the motion of a small glass bulb, of nearly the specific gravity of water, floating in the trough. A very slight difference of density between the water in *A* and *B*, gave motion to the bulb in *C*. The cock *D* being closed, the temperatures of *A* and *B* were adjusted so that one should be above and the other below that of maximum density. Having recorded the temperatures, *D* was opened, and any difference of density would be shown by a motion of the bulb towards the denser column. By carefully adjusting the temperatures, so that upon opening *D*

Fig. 300.



no motion in the trough C should result, a pair of temperatures was obtained, corresponding to the same density. From a series of such pairs, the differences of which were made successively smaller, Joule fixed the temperature of maximum density at 39.1° F. or 3.94° C., very nearly.

507. Radiation of Heat.—*Radiation* of heat is the communication of the vibrations of the heated body to the ether surrounding it, by which the waves of heat are transmitted in the manner already explained in the article Light. Heat rays differ from rays of light only in wave length, and are capable of reflection, refraction, interference, and polarization. A body not hot enough to send forth rays affecting the optic nerve, still sends out heat rays, nor can any body be so cold as not to radiate heat at all.

The intensity of heat radiated from a given source, is governed by the three following laws:

1. *The intensity of radiated heat varies as the temperature of the source.*

2. *It varies inversely as the square of the distance.*

3. *It grows less, while the inclination of the rays to the surface of the radiant grows less.*

The truth of these laws is ascertained by a series of careful experiments. But the second may be proved mathematically from the fact of propagation in straight lines, as in sound and light. For the heat, as it advances in every direction from the radiant, is spread over spherical surfaces which increase as the squares of the distances; therefore the intensities must grow less in the same ratio; that is, the intensities vary inversely as the squares of the distances.

The radiating power of a given body depends on the condition of its surface.

If a cubical vessel filled with hot water have one of its vertical sides coated with lamp black, another with mica, a third with tarnished lead, and the fourth with polished silver, and the heat radiated from these several sides be concentrated upon a thermometer bulb, the ratio of radiation will be found nearly as follows:

Lamp black	100	Tarnished lead	45
Mica	80	Polished silver	12

Polished metals generally radiate feebly; and this explains the familiar fact that hot liquids retain their temperature much better in bright metallic vessels than in dark or tarnished ones.

When the temperature of a body is gradually raised, not only

are new kinds of radiations produced, whose wave lengths are smaller than those already emitted, but the intensity of existing radiations also increases. A white-hot body emits more red rays than a red-hot body, and more non-luminous rays than a non-luminous body.

508. Equalization of Temperature.—Radiation is going on continually from all bodies, more rapidly in general from those most heated; and therefore there is a constant tendency toward an equal temperature in all bodies. A system of exchange goes on, by which the hotter bodies grow cool, and the colder ones grow warm, till the temperature of all is the same. But this equality does not check the radiation; it still goes forward, each body imparting to every other as much heat as it receives from it, the radiations emitted and absorbed by either body being equal not only in total heating effect, but being the same in the intensity, wave length, and plane of polarization of every component part of either radiation.

509. Reflection of Heat.—When rays of heat meet the surface of a body, some of them are *reflected*, passing off at the same angle with the perpendicular on the opposite side. But others pass *into* the body, and are said to be absorbed by it. It is true of waves of heat as of all other kinds of vibration, that when they meet a new surface and are reflected, the angle of incidence equals the angle of reflection, and that their intensity after reflection is weakened.

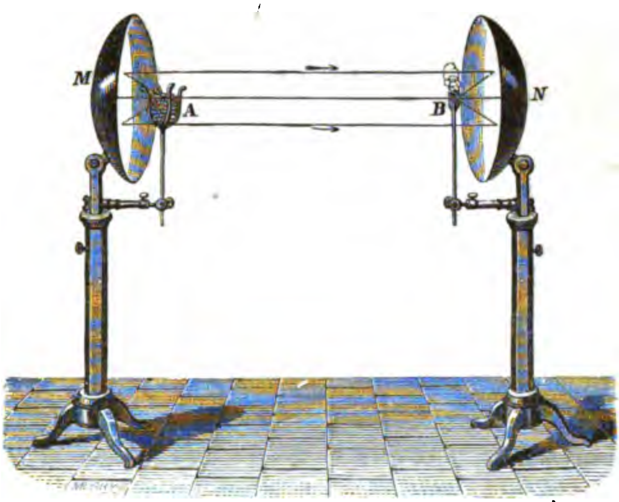
If a person, when near a fire, holds a sheet of bright tin so as to see the light of the fire reflected by it, he will plainly perceive that heat is reflected also. And if any *sound* is produced by the fire, as the crackling of combustion, or the hissing of steam from wood, the reflection of the sound is likewise heard. This simple experiment proves that waves of sound, of heat, and of light, all follow the same law of reflection.

510. Heat Concentrated by Reflection.—Let two polished reflectors, M and N (Fig. 301), having the form of concave paraboloids, be placed ten or fifteen feet apart, with their axes in the same straight line, and let a red-hot iron ball be in the focus A of one, and an inflammable substance, as phosphorus, in the focus B of the other; then the latter will be set on fire by the heat of the ball. The rays diverging from A to M are reflected in parallel lines to N, and then converged to B.

If, instead of phosphorus, the bulb of a thermometer is put in the focus B, a high temperature is of course indicated on the scale. Now remove the hot ball from A, and put in its place a lump of

ice; then the thermometer at B sinks far below the temperature of the room. This last experiment does not prove that *cold* is reflected as well as heat, but confirms what was stated (Art. 508),

FIG. 301.



that all objects radiate to one another till their temperatures are equalized. The ice radiates only a little heat, which is reflected to the thermometer, but the latter radiates much more, which is reflected to the ice, so that the temperature of the thermometer rapidly sinks.

511. Absorption of Heat.—The radiant heat which falls on a body and is not reflected or transmitted, is absorbed. The absorbing power in a body is found to be in general equal to its radiating power. It is very noticeable that bodies equally exposed to the radiant heat of the sun or a fire, become very unequally heated. A white cloth on the snow, under the sunshine, remains at the surface; a black cloth sinks, because it absorbs heat, and melts the snow beneath it. Polished brass before a fire remains cold; dark, unpolished iron, is soon hot.

Lamp black reflects little of the radiation which falls on it; nearly the whole is absorbed.

Polished silver reflects the greater part of the radiations falling upon it, absorbs only about 2½ per cent., and transmits none.

Rock salt reflects less than 8 per cent. of the radiation it receives, absorbs almost none, and transmits 92 per cent.

512. Diathermancy.—Substances which transmit heat rays, without themselves becoming hot, are called *Diathermanous;*

those which are heated by the transmission are said to be *Athermanous.*

Radiant heat passes freely through the atmosphere as well as through vacant space. The air is therefore said to be *diathermal;* it is also transparent, since it permits light to pass freely through it. But there are substances which allow the free transmission of the waves of light, but not those of heat; and there are others through which waves of heat can freely pass, but not those of light.

Water and glass, which are almost perfectly transparent to the faintest light, will not transmit the vibrations of heat unless they are very intense. If an open lamp-flame shines upon a thin film of ice, while nearly the whole of the *light* is transmitted, only 6 per cent. of the *heat* can pass through.

A plate of rock salt, one-tenth of an inch thick, will, as shown in the last paragraph, transmit 92 per cent. of the heat of a lamp; and if it be coated with lampblack so thick as to stop light completely, the heat is still transmitted with almost no diminution.

Prisms and lenses of rock salt have been used in illustrating refraction of heat, just as glass prisms and lenses are used in the case of luminous rays.

CHAPTER III.

SPECIFIC HEAT.—CHANGES OF CONDITION.—LATENT HEAT.

513. Specific Heat.—The energy of a single molecule of any chemical element, at a given temperature, is the same as for a molecule of any other element at the same temperature. To raise the temperature requires an addition of energy from an external source. Molecules of different substances have different weights. Hence a gram of light molecules would have more energy at a given temperature than a gram of heavier ones at the same temperature, for there would be a greater number of the lighter molecules. The amount of heat energy which must be supplied to one gram of a substance, that its temperature may be raised from 0° to 1° C., is termed the *specific heat* of that substance.

The quantity of heat or heat energy is measured in *gram calories.* A calorie is the heat necessary to raise the temperature of a gram of water from 0° to 1° C.

The specific heats of nearly all solids and liquids increase with temperature.

The mean specific heats of a few substances between 0° and 100° C. are given below to show how greatly they differ.

Water	1.000	Silver	0.057
Glass	0.190	Tin	0.056
Iron	0.113	Mercury	0.034
Nickel	0.110	Gold	0.032
Copper	0.094	Lead	0.032

The specific heat of water is greater than that of any other known substance, and therefore it is made the standard of comparison. The great specific heat of water moderates the changes of temperature upon islands and upon the sea-coast.

If a body expands by rise of temperature, part of the heat energy supplied to it is used in performing internal and external work—the internal work of separating the molecules and the external work against atmospheric pressure. The specific heat of a gas, which is confined to a *constant volume* during a rise of temperature (and must thus suffer an increase of pressure), is less than if allowed to expand, *i.e.*, at *constant pressure*. With constant volume no external or internal work is performed, and thus less heat is required to raise the temperature. These two specific heats for gases bear the ratio of 1.41 : 1. This ratio is the constant inserted in the formula for the velocity of sound (Art. 284).

514. Method of Finding Specific Heat.—The following is one of several methods of finding the specific heat of a substance: it is called the *method of mixtures*.

Let 100 grams of mercury at 100° C. be poured into 100 grams of water at 0° C., and suppose the temperature of the mixture to be 3.2° C. Let $x =$ the specific heat of mercury. Now 100 grams of water has been raised from 0° to 3.2°, requiring for this change 320 calories. This heat has been furnished by the mercury in falling from 100° to 3.2°. This is equal then to $100 \times x \times 96.8$. Now, as no heat has been lost, the calories received by the water are equal to those given up by the mercury or $320 = 9680\ x$, from which we find $x = 0.033$.

The specific heats of substances are also found by determining the amounts of ice at 0° C., or 32° F., which they will melt in cooling from a given temperature to that of melting ice.

The specific heat of a substance in a liquid state is generally greater than in the solid form. The specific heats of the more perfect gases are nearly equal to that of air, which is 0.237.

Ex. 1. A coil of copper wire weighing 45.1 grams was dropped into a calorimeter containing 52.5 grams of water at 10°. The copper before immersion was at 99.6° C., and the common temperature of copper and water after immersion was 16.8° C. Find the specific heat of the copper wire.

515. Apparent Conduction Affected by Specific Heat.—The conducting power of different substances cannot be correctly compared, without making allowance for their specific heat (Art. 501). For the heat which is communicated to one end of a rod, will collect at the other end more slowly, if a great share of it disappears on the way. For instance, at the same distance from the source of heat, wax is melted quicker on a rod of bismuth than on one of iron, though iron is the better conductor, because the specific heat of iron is three times as great as that of bismuth; the heat actually reaches the wax soonest through the iron, but not enough to melt it, because so much is required to raise the iron to a given temperature.

516. Changes of Condition.—Among the most important effects produced by heat are the changes of condition from solid to liquid and from liquid to gas, or the reverse, according as the temperature of a body is raised or lowered. Increase of heat changes ice to water, and water to steam, and the diminution of heat reverses these effects. A large part of the simple substances, and of compound ones not decomposed by heat, undergo similar changes at some temperature or other; and probably it would be found true of all if the requisite temperature could be reached.

The *melting-point* (called also *freezing-point*, or *point of congelation*) of a substance is the temperature at which it changes from a solid to a liquid, or the reverse.

The *boiling-point* is the temperature at which it changes from a liquid to a gas, or the reverse.

517. Latent Heat.—Whenever a solid becomes a liquid, or a liquid becomes a gas, a large amount of heat disappears, and is said to become *latent*. The heat-energy is expended in sundering the atoms, and perhaps in putting them into new relations and combinations, so that there is not the slightest increase of temperature after the change begins till it ends. The energy is not *lost*, but is treasured up in the form of *potential energy*, which becomes available whenever a change is made in the opposite direction. Using the heat-energy to turn water into steam, is like using the strength of the arm in coiling up a spring, or lifting a weight from the earth. The spring and the weight are each in

a condition to perform work. They have potential energy, which can be used at pleasure.

It has been already noticed that much heat disappears in bodies of great specific heat, as their temperature rises. But the amount which becomes latent, while a change of condition takes place, is vastly greater.

The number of calories necessary to transform one gram of water at 100° C. into steam at the same temperature is called the *latent heat of steam*, and equals 537. To transform n grams would, of course, take n times as many calories. Similarly the latent heat of water (or ice) is the number of calories necessary to transform one gram of ice at 0° into water at 0° C. The latent heat of ice is 80.

In steam boilers the pressure exerted by the steam upon the surface of the water has the effect of raising the water's boiling-point. The latent heat of the steam under such conditions is greater than the value given above. The determination of the energy stored up in steam at different temperatures and under different pressures is a problem of great technical importance. The total number of calories, Q, required to change n grams of water at 0° C. into steam at t° C. is expressed by the formula

$$Q = (606.5 + 0.305\,t)\,n.$$

For 100° C., $Q = 637\,n$; for 150° C., $Q = 652.2\,n$; for 200° C., $Q = 667.5\,n$.

518. Fusion or Melting.—The change from the solid to the liquid state may be either very gradual or very abrupt. As the temperature rises, many substances become pasty, like wrought iron at white heat, and for a considerable range of temperatures such substances are neither solid nor liquid, and no definite melting-point can be assigned. Ice passes very abruptly from the solid to the liquid state, probably during a rise of temperature not greater than 0.1° C.

From the beginning of fusion till the end of the change of condition there is no rise of temperature, the heat which does internal work being termed *latent heat of fusion*.

The latent heat of fusion of ice has already been given (Art. 517). The melting-points of a few substances are given below:

Ice	0° C.	Tin	235° C.
Spermaceti	49° C.	Lead	325° C.
White Wax	65° C.	Silver	1000° C
Sulphur	111° C.	Iron	1500° C.

The melting-point of a substance which expands on solidifying is lowered by great increase of pressure above the ordinary

pressure of the atmosphere, while that of a substance which contracts in solidifying is raised. The melting point of wax was raised from 65° C. to 75° C. by a pressure of 520 atmospheres, while the melting point of ice is lowered about 0.0074° C. for every additional pressure of one atmosphere.

Alloys are generally more fusible than the metals of which they are composed.

519. Vaporization.—The change from the liquid to the gaseous state is termed *vaporization*. This change is sometimes effected quietly without the formation of bubbles, then termed *evaporation*, and sometimes in a violent manner with the formation of bubbles, to which action the term *ebullition* is applied.

Vaporization is more rapid as the pressure upon the surface of the liquid is diminished.

The boiling point of water at one atmosphere, at the level of the ocean, is 100° C.; but upon the tops of high mountains the boiling point is 90° and 85° C., and in the air pump vacuum it is as low as 23° C.

The effect of diminished pressure to lower the boiling point is well shown by the following familiar experiment: In a thin glass flask, boil a little water, and after removing it from the fire, cork and invert the flask. The steam which is formed will soon press so strongly upon the water as to stop the boiling. When this happens, pour a little cold water upon the flask; the water within will immediately commence boiling violently, because the vapor is condensed and the pressure removed. This effect may be reproduced several times before the water in the flask is too cool to boil in a vacuum.

520. Other Causes Affecting the Boiling Point.—The boiling point is raised by substances in solution, provided they are less volatile than the liquid in which they are dissolved.

Water saturated with common salt boils at 109° C., and when chloride of calcium replaces the salt the boiling point is raised to 179° C. Substances held in suspension, but not dissolved, have no effect upon the boiling point.

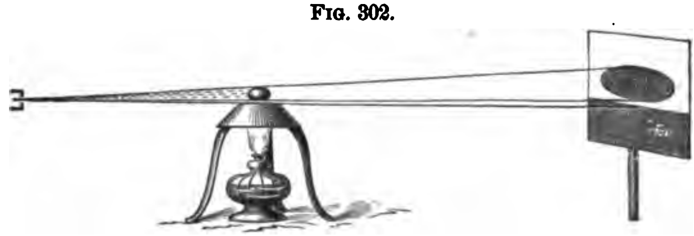
Water from which the dissolved air has been removed by previous ebullition, has been raised to 112° C. before boiling, the elastic air seeming to act as a spring to aid ebullition.

Water boils at a higher temperature in glass vessels than in metallic ones, rising as high as 105° C. before ebullition begins. If metal clippings or filings, or any angular fragments whatever which may serve as a nucleus, be dropped into the flask, the boiling point is brought down to 100° C., and the violent bumping which accompanies ebullition at the higher temperatures is prevented.

521. Spheroidal Condition.—When a little water is placed in a red-hot metallic cup, instead of boiling violently, and disappearing in a moment, as might be expected, it rolls about quietly in the shape of an oblate spheroid, and wastes very slowly. So drops of water, falling on the horizontal surface of a very hot stove, are not thrown off in steam and spray with a loud hissing sound, as they are when the stove is only moderately heated, but roll over the surface in balls, slowly diminishing in size till they disappear.

In such cases, the water is said to be in the *spheroidal state.* Not being in contact with the metal, it assumes the shape of an oblate spheroid, in obedience to its own molecular attractions and the force of gravity, as small masses of mercury do on a table. The reason why the water does not touch the hot metal is, that the heat causes a coat of vapor to be instantly formed about the drop, on which it rests as on an elastic cushion ; and as the vapor is a poor conductor of heat, further evaporation proceeds very slowly. It is easily seen that the spheroid does not touch the metal, by so arranging the experiment that a beam of light may shine horizontally upon the drop, and cast its shadow completely separated from that of the hot plate below it, as in Fig. 302.

FIG. 302.



If the heated surface is cooling, the temperature may become so low that the drop at length touches it, when in an instant violent ebullition takes place, and the water quickly disappears in vapor.

522. Evaporation.—Many liquids and even solids pass into the gaseous state by a slow and almost insensible process, which goes on at the *surface.* This is called *evaporation ;* and it takes place at all temperatures, but more rapidly as the temperature is higher. Ice and snow waste away gradually at temperatures far below 0° C., and the odor of brass, copper, and iron is attributed to an insensible evaporation of these metals.

523. Condensation.—The change from the condition of vapor to the liquid state is called *condensation.* This change of state may be caused by cooling and by compression. A saturated

vapor at any given temperature and pressure will be partially condensed by either lowering the temperature or by increasing the pressure. Those gases which have usually been called *permanent gases*, because, under ordinary conditions they are very far removed from their point of condensation, have been reduced to the liquid state by very low temperatures and great pressures combined.

Vapors give up their latent heat of vaporization during the process of condensation; the latent heat of steam may be determined by passing a known weight of steam at 100° C. into a given quantity of water at a known temperature, and taking the resulting temperature.

Suppose 10 grams of steam at 100° C. to be condensed by 60 grams of water at 0° C., and that the resulting temperature is 91° C. The 60 grams of water raised from 0° to 91° required $60 \times 91 = 5460$ heat units; 10 grams of steam, after condensation at 100°, gave up 90 of these 5460 units in cooling from 100° to 91°, leaving 537 to each gram of steam as the latent heat of its vaporization.

524. Solidification.—Substances which have been melted and which cool slowly while passing into the solid state usually assume a regular crystalline structure. If they expand on solidifying, the solid will float in the liquid, but if they contract the solid will sink.

A liquid may be cooled below its normal temperature of solidification. A hot saturated solution of Glauber's salt, cooled slowly and at rest, will remain liquid at the ordinary temperature of the atmosphere; but upon being suddenly jarred, or when a crystal of the salt is dropped into the liquid, the molecular equilibrium is destroyed and solidification ensues at once. Water which has been boiled, to free it from air, may be cooled to —10° C., or even lower, without freezing; but any vibration causes instant crystallization. In all such cases the latent heat of fusion becomes sensible, and may be felt by placing the hands upon the containing vessel.

The freezing point of water containing salt in solution is lower than that of pure water. Sea water freezes at —2.5° C. to —3° C.; the ice is pure, containing none of the salt.

525. Freezing Produced by Melting.—Since a great amount of heat disappears in a substance as it passes from the solid to the liquid state, the loss thus occasioned may produce freezing in a contiguous body. When salt and powdered ice are mixed, their union causes liquefaction. And if this mixture is surrounded by bad conductors, and a tin vessel containing some

liquid be placed in the midst of it, the latter is frozen by the abstraction of heat from it, by the melting of the ice and salt. In this way ice creams and similar luxuries are easily prepared in hot as well as in cold weather.

526. Freezing by Evaporation.—In like manner, freezing by evaporation is explained. Put a little water in a shallow dish of thin glass, and set it on a slender wire-support under the receiver of an air pump. Beneath the wire-support place a broad dish containing sulphuric acid. When the air is exhausted, the water in a few moments is found frozen. As the pressure of the air is taken off, evaporation proceeds with increased rapidity, and the requisite heat for this change of condition can be taken only from the dish of water. But the atmosphere of vapor retards the process by its pressure ; hence the sulphuric acid is placed in the receiver, so as to seize upon the vapor as fast as formed, and thus render the vacuum more complete. The water is frozen by giving up its heat to become latent in the vapor, so rapidly formed ; but when this vapor becomes liquid again in combining with the acid, the same heat reappears in raising the temperature of the acid.

Thin cakes of ice may sometimes be procured, even in the hottest climates, by the evaporation of water in broad shallow pans under the open sky, where radiation by night aids in reducing the temperature. The pans should be so situated as to receive the least possible heat by conduction.

Various ice-making machines have been devised in which the vaporization of some volatile liquid, such as ether, liquid ammonia, liquid sulphurous acid, &c., abstracts sufficient heat from the water to freeze it.

527. Regelation.—If two pieces of ice at 0° C., having smooth surfaces, be pressed together, they will soon adhere, and will do this in air, in water, or in vacuo. This freezing together again is called *regelation.*

The interior of a block of melting ice is a little colder than the surface : now when the two surfaces are pressed together, the very thin film of water which covers them is removed from the warmer air, and is in the same condition as though transferred to the interior of a block, the lower temperature of which freezes it.

CHAPTER IV.

TENSION OF VAPOR.—THE STEAM-ENGINE.—MECHANICAL EQUIVALENT OF HEAT.

528. Dalton's Laws.—

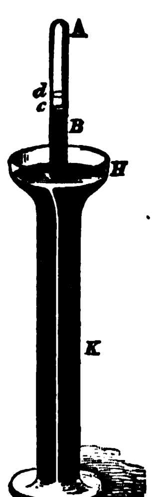
1. Whatever be the temperature of a liquid which partly fills a vessel, vaporization will go on till the vessel is filled with vapor, of a density determined solely by the temperature, after which vaporization will cease.

2. If the space occupied by the vapor be made larger, the temperature being the same, then vaporization will again go on till the density is the same as before. If the space be made smaller, the temperature remaining constant, a part of the vapor returns to the liquid state, and the remaining vapor will have the same density as before.

3. If, besides the liquid and its vapor, the vessel contains any gas, *not capable of chemical action on the liquid,* then exactly the same amount of vapor, of the same density as before, will be formed; but the time required to reach the maximum density will be greater because of the mechanical obstruction to a rapid diffusion, which the gas offers.

A vapor at the maximum density and pressure for the given temperature is called a *saturated* vapor.

FIG. 303.



529. Experimental Illustration.—Fill a barometer tube *A B* (Fig. 303) full of mercury; close the open end with the finger and invert into the cup *H* of the deep mercury cistern *H K*. With a pipette, the tube of which is bent upwards at the end, transfer enough ether to the barometer tube to leave a thin film of liquid *c d*, after the space *A d* is filled with saturated vapor. Measure the height *c H* of the mercury column. If the tube *A B* be raised, tending to increase the space *A d* above the liquid, more vapor will form and *c H* will remain unaltered; if the tube *A B* be depressed, tending to diminish the space *A d*, vapor will condense to liquid again, and *c H* will still be unaltered.

To show the effect of change of temperature use a barometer

tube bent at its closed end as in Fig. 304, so that a portion of the bend may either be surrounded with cooling mixtures, as at *A*, or may be warmed by a flame. Upon raising the temperature of the contained vapor its tension will increase and the mercury column *C B* will be shortened; upon lowering the temperature the tension will decrease, and *C B* will lengthen as the mercury rises.

FIG. 304.

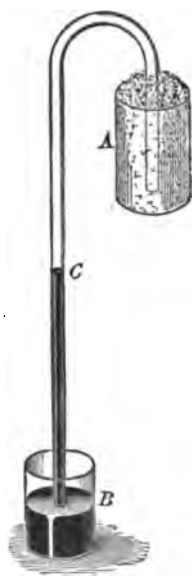
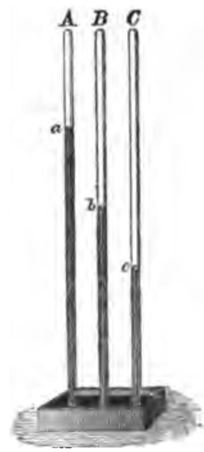


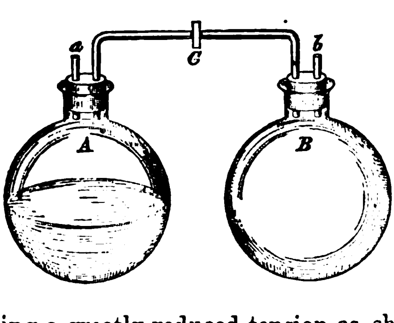
FIG. 305.



530. Tensions of Different Vapors. — Transfer to three barometer tubes, *A*, *B*, and *C* (Fig. 305), water, alcohol and ether respectively, and the mercury columns will stand at different heights *a*, *b* and *c*, showing that the tensions of the three vapors are not the same at the same temperature.

531. Tension in Generator and Condenser.— Let two vessels, *A* and *B* (Fig. 306), be connected by a pipe furnished with a stop-cock *C*. Let the tubes *a* and *b* be connected with separate manometers to indicate the tensions of the vapor in *A* and *B* when *C* is closed. Having partly filled *A* with water, cause it to boil until all air has been driven from both flasks through *C* and the loosened stopper of *B*; now close *B* and remove the lamp. The two manometers will indicate the same tension in both flasks.

FIG. 306.



Now close *C* and surround *B* with cold water; part of the vapor in it will be condensed, the remainder having a greatly reduced tension as shown by the fall of the mercury column in its manometer. Apply heat to *A*, thus forming new vapor of higher tension than before, as will be shown by its manometer reading.

If now C be opened the manometer connected with A will fall to the same reading as that of B, and the two will indicate this same reading just as long as the temperature of B is kept constant and below that of A.

The liquid in A merely distills over to B, at the tension of the vapor in the colder vessel.

532. Heat Energy in Steam.—It has been already noticed that while water is heated, and especially after it is converted into steam by boiling, the heat apparently lost is so much energy treasured up ready for use, as truly as when energy is expended in lifting great weights, which by their descent can do the work desired. In modern engineering, the energy of steam is employed more extensively, and for more varied purposes, than any other. Every steam-engine is a machine for transforming the internal motion of heated steam into some of the visible forms of motion.

533. Tension of Steam.—When steam is formed by boiling water in the open air, its tension is equal to that of the air, and therefore ordinarily about fifteen pounds to the square inch. But when it is formed in a tight vessel, so that it cannot expand, as the temperature of the water is raised the tension is increased in a much greater ratio; because the same steam has greater tension at a higher temperature, and besides this, new steam is continually added.

The following table gives the temperature corresponding to various atmospheres of tension:

Atmospheres,	1	2	4	5	9	10	14	15	19	20
Temperature C.°,	100	120.6	144	152.2	175.8	180.3	195.5	198.8	210.4	213

534. The Steam-Engines of Savery and Newcomen.—The only steam-engines that were at all successful before the great improvements made by Watt, were the engine of Savery and that of Newcomen. No other purpose was proposed by either than that of removing water from mines.

In the engine of Savery, steam was made to raise water by acting on it directly, and not through the intervention of machinery.

It consisted of a boiler B (Fig. 307); a cylinder A, with a valve at c opening inward, and one at d opening outward; a pipe e to discharge cold water upon the cylinder, and a steam pipe f, from the boiler to the cylinder.

First the steam-cock at f is opened and steam fills the cylinder A, driving the air out through the valve d. Next f is closed and

the cock *e* is opened, allowing cold water to flow over the cylinder from the delivery pipe *O*, thus condensing the steam in *A*, and creating a vacuum, into which the atmosphere forces water from the supply *P*, through the valve *c*. Now *e* is closed and *f* opened

FIG. 307.

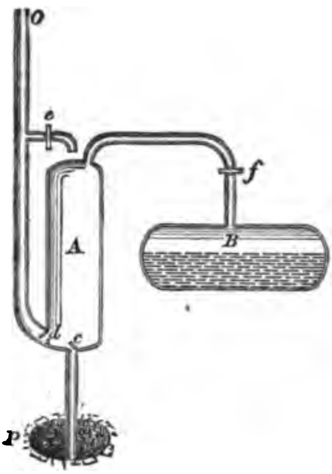
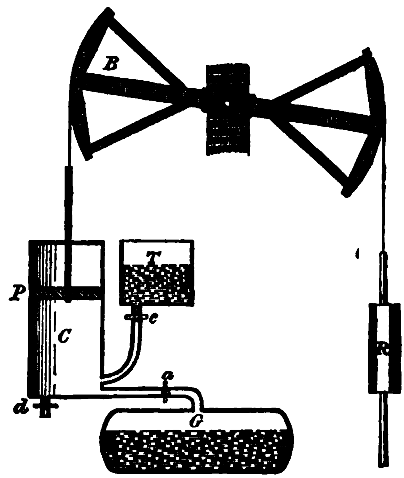


FIG. 308.



again, and steam enters the cylinder *A* and drives the water out through *d*. When *A* is full of steam the operation is repeated as before.

Newcomen used steam to work a common pump. The weighted pump rod *R* (Fig. 308) was attached to one end of a working beam *B*, while at the other end of *B* was hung the piston *P*, working steam tight in the cylinder *C*. Steam at atmospheric pressure from the boiler *G* enters *C* through the cock *a*, and *P* being pressed upon equally on both sides is drawn to the top of *C* by the weight of the pump rod *R*. Now *a* is closed and *e* is opened, permitting cold water from a tank *T* to flow into *C*, which condenses the steam, creating a vacuum, and allows the piston to descend under atmospheric pressure. When *P* has reached the bottom of *C*, *e* having been closed, *a* and *d* are opened and steam enters the cylinder, while the injection water from *T* flows out through *d*, and the piston *P* rises as at first. On closing *a* and *d* and opening *e* the stroke is repeated.

As the water was raised by the direct pressure of the atmosphere, this invention of Newcomen was called the *atmospheric engine*.

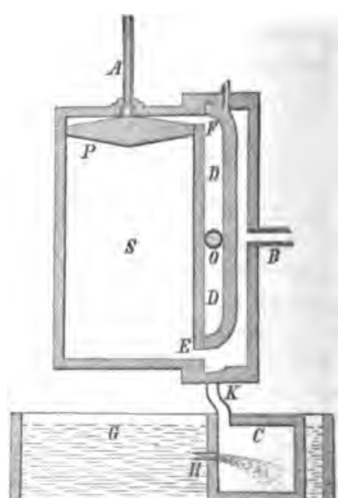
In these diagrams of Savery's and Newcomen's engines, all details of valves or other working parts have been omitted, that the principle alone might claim attention.

In neither of these methods was steam used economically as a power. The movements in both cases were sluggish, and a large part of the energy was wasted, because the steam was compelled to act upon a cold surface, which condensed it before its work was done.

535. The Steam-Engine of Watt.—Steam did not give promise of being essentially useful as a power till Watt, in the year 1760, made a change in the atmospheric engine, which prevented the great waste of force. Newcomen introduced the cold water which was to condense the steam into the steam cylinder itself; and the cylinder must be cooled to a temperature below 100° F., else there would be steam of low tension to retard the descent of the piston. But when the piston was to be raised, the cylinder must be heated again to 212° F., in order that the admitted steam might balance the pressure of the air.

In the engine of Watt, the steam is condensed in a separate vessel called the *condenser*. The steam cylinder is thus kept at the uniform temperature of the steam. In the first form which he gave to his engine, he so far copied the atmospheric engine as to allow the piston, after being pressed down by steam, to be raised again by the load on the opposite end of the great beam, while the steam circulates freely below and above the piston. This was called the *single-acting* engine, and might be successfully used for the only purpose to which any steam-engine was as yet applied, namely, pumping water from mines. But he almost immediately introduced the change by which the whole force of the steam was brought to act on the upper and the under side of the piston. It thus became *double-acting*, and the steam force was no longer intermittent.

FIG. 309.

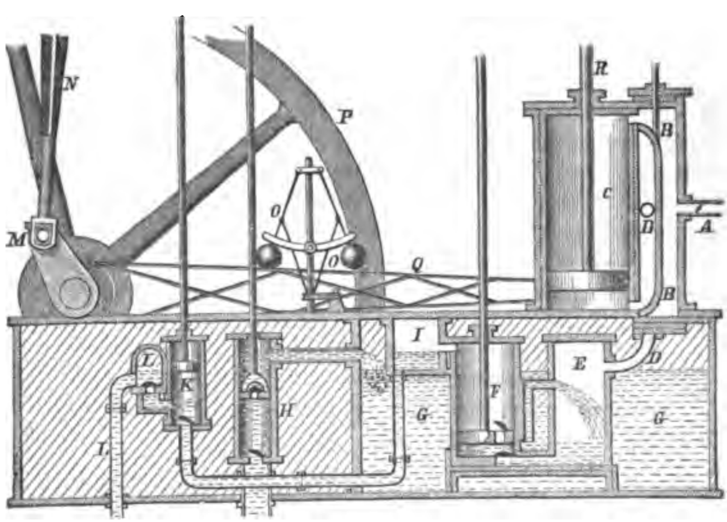


536. The Double-acting Engine.—Let S (Fig. 309) be the steam cylinder, P the piston, A the piston rod, passing with steam-tight joint through the top of the cylinder, C the condenser, kept cold by the water of the cistern G, B the steam pipe from the boiler, K the eduction pipe, which opens into the valve chest at O, D D the D-valve,

E F the openings from the valve-chest into the cylinder. As the D-valve is situated in the figure, the steam can pass through B and E into the cylinder below the piston, while the steam above the piston can escape by F through O and K to the condenser, where it is condensed as fast as it enters; so that in an instant the space above the piston is a vacuum, while the whole force of the steam is exerted on the under side. The piston is therefore driven upward without any force to oppose it. But before it reaches the top, the D-valve, moved by the machinery, begins to descend, and shut off the steam from E and admit it to F, and, on the other hand, to shut F from the eduction pipe O, and open E to the same. The steam will then press on the top of the piston, and there will be a vacuum below it, so that the piston descends with the whole force of the steam, and without resistance. To render the condensation more sudden, a little cold water is thrown into the condenser at each stroke through the pipe H.

537. Condensing Engine.—The principle of the condensing engine is illustrated by the figure and description of the preceding article. But the condensing apparatus of this kind of engine requires many other parts, most of which are presented in Fig. 310. C is the steam cylinder; R the rod connecting its

FIG. 310



piston with the end of the working beam, not represented; A the steam-pipe and throttle-valve; B B the D-valve; D D the eduction pipe, leading from the valve-chest to the condenser E; G G

the cold water surrounding the condenser; *F* the air-pump, which keeps the condenser clear of air, steam, and water of condensation; *I* the hot well, in which the water of condensation is deposited by the air-pump; *K* the hot-water pump, which forces the water in the hot well through *L* to the boiler; *H* the cold-water pump, by which water is brought to the cistern *G G*; the rods of all the pumps, *F*, *K*, and *H*, are moved by the working beam; *P* the fly-wheel; *M* the crank of the same, *N* the connecting-rod, by which the working beam conveys motion to the fly-wheel; *Q* the excentric rod, by which the D-valve is moved; *O O* the governor, which regulates the throttle-valve in the steam-pipe *A*.

There is much economy of fuel and saving of wear in the machinery, arising from the proper adjustment of the valves. If the steam enters the cylinder during the whole length of a stroke of the piston, its motion is *accelerated;* and is therefore swiftest at the instant before being stopped; thus the machinery receives a violent shock. If the valve is adjusted to *cut off* the steam when the piston has made one-third or one-half of its stroke, the diminishing tension may exert about force enough, during the remaining part, to keep up a uniform motion. The *cut-off*, however, should be regulated in each engine, according to friction and other obstructions.

538. Non-condensing Engine.—For many purposes, especially those of locomotion, it is advantageous to dispense with the large weight and bulk of machinery necessary for condensation, and do the work with steam of a higher tension. If (Fig. 310) the condenser, cistern, and all the pumps are removed, then the steam is discharged from *E* and *F* at each stroke into the air. Therefore the steam in that part of the cylinder which is open to the air, will have a tension of 15 lbs. per inch; and, consequently, the steam on the opposite side of the piston must have a tension 15 lbs. per inch greater than before, in order to do the same work.

Steam of a pressure not greater than 45 lbs. per inch (above the atmosphere) is called *low pressure* steam, or *low steam; high steam* is at a pressure above this, and not uncommonly runs higher than 200 lbs. per inch by the gauge.

539. Calculation of Steam Power.—Assuming that the pressure within the cylinder of a steam-engine remains constant throughout the whole of the stroke, we can find the horse-power developed in each cylinder of an engine, having given—

A = area of piston in square inches.

P = pressure upon the piston in pounds per square inch.
S = length of stroke in feet.
R = number of revolutions per minute.

Here P denotes the *intensity* of the pressure on the piston in pounds weight per square inch. The total pressure on the piston is the weight of $A\ P$ pounds. This is the acting force, and the distance through which it moves in each stroke is S feet.

Thus the work done in each stroke is $S\ A\ P$ foot-pounds.

Since there are two strokes for each revolution, the number of strokes per minute is $2\ R$, and the work done per minute is $2\ S\ R\ A\ P$ foot-pounds. Thus the horse-power developed is

$$\text{H. P.} = 2\ S\ R\ A\ P/33{,}000.$$

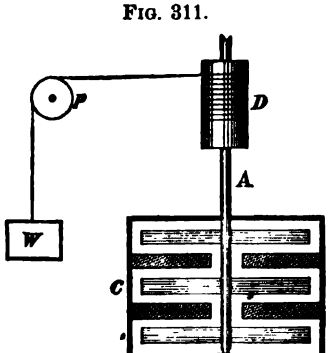
540. Mechanical Equivalent of Heat.—We have seen that a given quantity of heat represents a definite amount of energy, and accordingly a gram calorie must be equivalent to a certain number of ergs (Art. 33). Joule, Rowland, and others have performed experiments which demanded the utmost accuracy and patience, and have determined that one calorie is equivalent to 416×10^5 ergs. This coefficient is called the *mechanical equivalent of heat.* Joule, who made the first determination, used the foot-pound and the pound-degree-Fahrenheit as units of work and heat respectively. The mechanical equivalent in these units may be expressed by the following statement:

The energy required to heat one pound of water one degree F., is equal to that which would lift 772 pounds the vertical distance of one foot, or is equal to 772 foot-pounds.

Joule's mode of determining this value of the mechanical equivalent is the following:

A weight W (Fig. 311), by means of a cord passing over a pully p and around a drum D, gives to the vertical axis A a rapid rotation. Attached to this axis are a number of radial arms, or paddles, as shown in the figure; projecting from the sides of the cylinder C, in which these arms rotate, are fixed arms, as shown, to arrest any tendency to a rotary motion of the water in the cylinder.

FIG. 311.



If one pound of water at 60° F. be put into the cylinder C, it will require the expenditure of 772 foot-pounds of energy on the part of the falling weight W to raise its temperature by agitation to 61° F.

Heat is the lowest form of energy. When by any action energy is liberated which is not specially directed by circumstances, it takes the form of heat. The energy of a dynamo-electric machine, if not made to do outside work, transforms itself into heat in the circuit. All energy tends to assume the form of heat, and it may be expected that the total energy of the universe will ultimately turn into heat, as was stated in Art. 37.

CHAPTER V.

TEMPERATURE OF THE ATMOSPHERE. — MOISTURE OF THE ATMOSPHERE. — DRAUGHT AND VENTILATION.

541. Manner in which the Air is Warmed.—The space through which the earth moves around the sun is intensely cold, probably 75° below zero; and the one or two hundred miles of height occupied by the atmosphere is too cold for animal or vegetable life, except the lowest stratum, three or four miles in thickness. This portion receives its heat mainly by convection. The radiated heat of the sun passes through the air, warming it but little, and on reaching the earth is partly absorbed by it. The air lying in contact with the earth, and thus becoming warmed, grows lighter and rises, while colder portions descend and are warmed in their turn. So long as the sun is shining on a given region of the earth, this circulation is going on continually.

542. Limit of Perpetual Frost.—At a moderate elevation, even in the hottest climate, the temperature of the air is always as low as the freezing-point. Hence the permanent snow on the higher mountains in all climates. The limit at the equator is about three miles high, and, with many local exceptions, it descends each way to the polar regions, where it is very near the earth. The descent is more rapid in the temperate than in the torrid or frigid zones.

543. Isothermal Lines.—These are imaginary lines on each hemisphere, through all those points whose mean annual temperature is the same. At the equator, the mean temperature is about 82° F., and it decreases each way toward the poles, but not equally

on all meridians. Hence the isothermal lines deviate widely from parallels of latitude. Their irregularities are due to the difference between land and water, in absorbing and communicating heat, to the various elevations of land, especially ranges of mountains, to ocean currents, &c. In the northern hemisphere, the isothermal lines, in passing westward round the earth, generally descend toward the equator in crossing the oceans, and ascend again in crossing the continents. For example, the isothermal of 50° F., which passes through China on the parallel of 44°, ascends in crossing the eastern continent, and strikes Brussels, lat. 51°; and then on the Atlantic, descends to Boston, lat. 42°, whence it once more ascends to the N. W. coast of America. The lowest mean temperature in the northern hemisphere is not far from zero, but it is not situated at the north pole. Instead of this, there are *two* poles of greatest cold, one on the eastern continent, the other on the western, near 20° from the geographical pole. There are indications, also, of two south poles of maximum cold.

544. Moisture of the Atmosphere.—By the heat of the sun all the waters of the earth form above them an atmosphere of vapor, or invisible moisture, having more or less extent and tension, according to several circumstances. Even ice and snow, at the lowest temperatures, throw off some vapor.

At a given temperature, there can exist an atmosphere of *vapor* of the same height and tension, whether there is an atmosphere of *oxygen* and *nitrogen* or not (Art. 528). Vapor is not *suspended* in the air, or *dissolved* by it, but exists independently. And yet it is by no means always true that there is actually the same tension of vapor as there would be if it existed alone, because of the time required for the formation of vapor, on account of mechanical obstruction presented by the air; whereas, if no air existed, the vapor would form almost instantly.

545. Temperature and Tension of Vapor.—The degree of tension of vapor forming without obstruction, depends on its temperature, but varies far more rapidly, increasing pretty nearly in a geometrical ratio, while the heat increases arithmetically; thus the tension at 212° F. is 1 atmosphere, at 249° F. is 2 atmospheres, at 291° F. is 4 atmospheres, and at 339° F. is 8 atmospheres. Hence, if vapor should receive its full increment of tension, while the thermometer rises 10 degrees from 80° to 90°, a vastly greater quantity would be added than when it rises 10 degrees from 40° to 50°. On the contrary, if vapor is at its full tension in each case, much more water will be precipitated in cooling from 90° to 80° than from 50° to 40°.

546. Dew-point.—This is the temperature at which vapor, in a given case, is precipitated into water in some of its forms. If there was no air, the dew-point would always be the same as the existing temperature; since lowering the temperature in the least degree would require a diminished tension or quantity of vapor, some must therefore be condensed into water. But in the air the tension may not be at its full height, and therefore the temperature may need to be reduced several degrees before precipitation will take place. A comparison of the temperature with the dew-point is one of the methods employed for measuring the humidity of the air.

547. Measure of Vapor.—The measure of the vapor existing at a given time, is expressed by two numbers, one indicating its *tension*,—*i. e.*, the height of the column of mercury which it will sustain; the other, *humidity*,—*i.e.*, its quantity per cent., as compared with the greatest possible amount at that temperature. Thus, tension = 0.6, humidity = 83, signifies that the quantity of vapor is sufficient to support six-tenths of an inch of mercury, and is 83 hundredths of the quantity which *could* exist at that temperature. The greatest tension possible at zero F., is 0.04; at the freezing point, 0.18; at 80° F., 1.0. At the lowest natural temperatures, the maximum tension is doubled every 12° or 14°; at the highest, every 21° or 22°.

548. Hygrometers. — This is the name usually given to instruments intended for measuring the moisture of the air. But the one most used of late years is called the *psychrometer*, which gives indication of the amount of moisture by the degree of *cold* produced in evaporation; for evaporation is more rapid, and therefore the cold occasioned by it the greater, according as the air is drier. The psychrometer consists of two thermometers, one having its bulb covered with muslin, which is kept moistened by the capillary action of a string dipping in water.

The wet-bulb thermometer will ordinarily indicate a lower temperature than the dry-bulb; if, in a given case, they read alike, the humidity is 100. The instrument is accompanied by tables, giving tension and humidity for any observation.

Various formulæ and complete tables may be found in the "Smithsonian Meteorological and Physical Tables."

549.—Dew.—Frost.—The deposition called *dew* takes place on the surface of bodies, by which the air is cooled below its dew-point. It is at first in the form of very small drops, which unite and enlarge as the process goes on. Dew is formed in the evening or night, when the surfaces of bodies exposed to the sky

become cold by radiation. As soon as their temperature has descended to the dew-point, the stratum of air contiguous to them deposits moisture, and continues to do so more and more as the cold increases.

Of two bodies in the same situation, that will receive most dew which radiates most rapidly. Many vegetable leaves are good radiators, and receive much dew. Polished metal is a poor radiator, and ordinarily has no dew deposited on it.

Sometimes, however, good radiators have little dew, because they are so situated as to obtain heat nearly as fast as they radiate it. Dew is rarely formed on a bed of sand, though it is a good radiator, because the upper surface gets heat by conduction from the mass below. Dew is not formed on water, because the upper stratum sinks and gives place to warmer ones.

Bodies most exposed to the open sky, other things being equal, have most dew precipitated on them. This is owing to the fact, that in such circumstances, they have no return of heat either by reflection or radiation. If a body radiates its heat to a building, a tree, or a cloud, it also gets some in return, both reflected and radiated. Hence, little dew is to be expected in a cloudy night, or on objects surrounded by high trees and buildings.

Wind is unfavorable to the formation of dew, because it mingles the strata, and prevents the same mass from resting long enough on the cold body to be cooled down to the dew-point.

When the radiating body is cooled below the freezing point, the water deposited takes the solid form in fine crystals, and is called *frost*. Frost will often be found on the best radiators, or those exposed to the open sky, when only dew is found elsewhere.

550. Fog.—This form of precipitation consists of very small globules of water sustained in the lower strata of the air. Fog occurs most frequently over low grounds and bodies of water, where the humidity is likely to be great. If air thus humid mixes with air cooled by neighboring land, even of less humidity, there will probably be more vapor than can exist at the intermediate temperature, for the reason mentioned in Art. 545. The case may be illustrated thus. Let two masses of air of equal volumes be mixed, the temperature of one being 40° F., the other 60° F., and each containing vapor at the highest tension. Then the mixture will have the mean temperature of 50°, and the vapor of the mixture will also be the *arithmetical* mean between that of the two masses. But, according to the law (Art. 545), the vapor can only have a tension which is nearly a *geometrical* mean between the two, and that is necessarily lower than the *arithmetical* mean; hence the excess must be precipitated. If 8 lbs. of vapor were in one volume

and 18 lbs. in the other, an equal volume of the mixture would have $\frac{1}{2}(8 + 18) = 13$ lbs. of moisture; but at the mean temperature of 50°, only $\sqrt{8 \times 18} = 12$ lbs. could exist as vapor; therefore *one pound* must be precipitated. And even if one of the masses had a humidity somewhat below 100, still some precipitation is likely to take place.

551. **Cloud.**—The same as fog, except at a greater elevation. Air rising from heated places on the earth, and carrying vapor with it, is likely to meet with masses much colder than itself, and depositions of moisture are therefore likely to take place. Mountain-tops are often capped with clouds, when all around is clear. This happens when lower and warmer strata are driven over them, and thus cooled below the dew-point. The same air, as it continues down the other side, takes up its vapor again, and is as transparent as it was before ascending. A person on the summit perceives a chilly fog driving by him, but the fog was an invisible vapor a few minutes before reaching him, and returns to the same condition soon after leaving him. The cloud *rests* on the mountain; but all the particles which compose it are swiftly *crossing over*. Clouds are often above the limit of perpetual frost; they then consist of crystals of ice.

552. **Rain, Mist.**—Whether the precipitated moisture has the form of cloud or rain, depends on the rapidity with which precipitation takes place. If currents of air are in rapid motion, if the temperature of masses, brought into contact by this motion, are widely different, and if their humidity is at a high point, the vapor will be precipitated so rapidly that the globules will touch each other, and unite into larger drops, which cannot be sustained. Globules of fog and cloud, however, are specifically as heavy as drops of rain; but they are sustained by the slightest upward movements of the air, because they have a great surface compared with their weight. A globule whose diameter is 100 times less than that of a drop of rain, meets with 100 times more obstruction proportionally, since the weight is diminished a million times $(\frac{1}{100})^3$, and the section only ten thousand times $(\frac{1}{100})^2$. So the dust of even heavy minerals is sustained in the air for some time, when the same substances, in the form of sand, or coarse gravel, fall instantly.

If a cloud of fine dust contains so much matter as to make the *mass* of a cubic foot of the dusty air greater than that of a cubic foot of pure air, it will descend. If the mean density of a fog is greater than that of the purer surrounding air, it will settle

down into hollows and valleys; if its mean density is less than the air, it will rise as cloud.

Mist is fine rain; the drops are barely large enough to make their way slowly to the earth.

553. Hail, Sleet, Snow.—When the air in which rapid precipitation occurs, is so cold as to freeze the drops, hail is produced. As hailstones are not usually in the spherical form when they reach the earth, it is supposed that they are continually receiving irregular accretions in their descent through the vapor of the air. Hail-storms are most frequent and violent in those regions where hot and cold bodies of air are most easily mixed. Such mixtures are rarely formed in the torrid zone, since there the cold air is at a great elevation; in the frigid zone, no hot air exists at any height; but in the temperate climates, the heated air of the torrid, and the intensely cold winds of the frigid zone, may be much more easily brought together; and accordingly, in the temperate zones it is that hail-storms chiefly occur. Even in these climates, they are not frequent except on plains and valleys contiguous to mountains which are covered with snow during the summer. The slopes of the mountain sides give direction to currents of air, so that masses of different temperature are readily mingled together.

Sleet is frozen mist, that is, it consists of very small hailstones.

Snow consists of the small crystals of frozen cloud, united in flakes. Like all transparent substances, when in a pulverized state, it owes its whiteness to innumerable reflecting surfaces. A cloud, when the sun shines upon it, is for the same reason intensely white (Art. 371).

554. Draught of Flues.—The effect of the sun's heat in causing circulation of the air has been already considered (Art. 268–272). Similar movements on a limited scale are produced whenever a portion of the air is heated by artificial means. Thus, the air of a chimney is made lighter by a fire beneath it, than a column of the outer air extending to the same height. It is therefore pressed upward by the heavier external air, which descends and moves toward the place of heat. The difference of weight in the two columns is greater, and therefore the draught stronger, if the chimney is high, provided the supply of heat is sufficient to maintain the requisite temperature. Chimneys are frequently built one or two hundred feet high for the uses of manufactories. The high fireplaces and large flues of former times were unfavorable for draught, both because much cold air could mingle with that which was heated, and because there was room for external air to descend by the side of the ascending column. For

good draught, no air should be allowed to enter the flue except that which has passed through the fire.

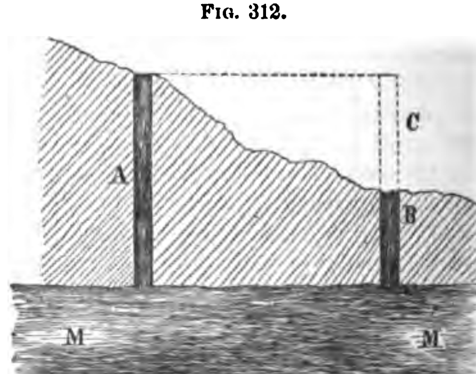
555. Ventilation of Apartments.—The air of an apartment, as it becomes vitiated by respiration, may generally be removed, and fresh air substituted, by taking advantage of the same inequality of weight in air-columns, which has been mentioned. If opportunity is given for the warm impure air to escape from the top of a room, and for external air to take its place, there will be a constant movement through the room, as in the flue of a chimney, though at a slower rate. If the external air is cold, the weight of the columns differs more, and therefore the ventilation is more easily effected. But in cold weather, the air, before being admitted to the room, is warmed by passing through the air-chambers of a furnace. When there is a chimney-flue in the wall of a room, with a current of hot air ascending in it, the ventilation is best accomplished by admitting the air into the flue at the upper part of the room; since it will then be removed with the velocity of the hot-air current.

The tendency of the air of a warm room to pass out near the top, while a new supply enters at the lower part, is shown by holding the flame of a candle at the top, and then at the bottom, of a door which is opened a little distance. The flame bends outward at the top and inward at the bottom.

The impure air of a large audience-room is sometimes removed by a mechanical contrivance, as, for instance, a fan-wheel placed above an opening at the top, and driven by steam.

The ventilation of mines is accomplished sometimes by a fire built under a shaft, fresh air being supplied by another shaft, and sometimes by a fan-wheel at the top of the shaft. If there happen to be two shafts which open to the surface at very different elevations, ventilation may be effected by the inequality of temperature which is likely to exist within the earth and above it. Let *M M* (Fig. 312) be the vertical section of a mine through two shafts *A* and *B*, which open at different heights to the surface of the earth. If the

FIG. 312.



external air is of the same temperature as the air within the earth, then the column A in the longer shaft has the same weight as B and C together, measured upward to the same level. In that case, which is likely to occur in spring and fall, there is no circulation without the use of other means. But in summer the air C is warmer than A and B; therefore A is heavier than $B + C$. Hence there is a current of air down A and up B. In winter, C is colder than air within the earth; therefore $B + C$ are together heavier than A, and the current sets in the opposite direction, down B and up A.

556. Sources of Heat.—*The sun*, although nearly a hundred millions of miles from the earth, is the source of nearly all the heat existing at its surface. The interior of the earth, except a thickness of forty or fifty miles next to the surface, is believed to be in a condition of heat so intense that all the materials composing it are in the melted state. But the earth's crust is so poor a conductor that only an insensible fraction of all this heat reaches the surface.

The energy radiated to the earth by the sun amounts to 83 foot-pounds per square foot of the earth's surface per second. Sir William Thomson, in his memoir on the "Mechanical Value of a Cubic Mile of Sunlight," says that the energy of the waves comprised within a cubic mile of ether near the earth is equal to about 12,050 foot-pounds.

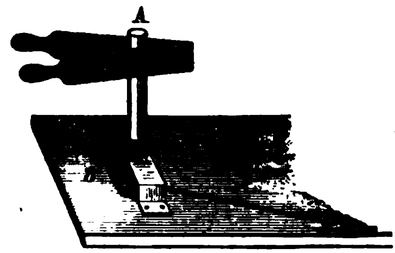
Mechanical operations are always attended by a development of heat. For example, if a broad surface of iron were made to revolve, rubbing against another surface, nearly all the energy expended in overcoming the friction would appear as heat, a comparatively small part being conveyed through the air as sound. The cutting tool employed in turning an iron shaft has been known to generate heat enough to raise a large quantity of cold water to the boiling-point, and to keep it boiling for an indefinite time. It is a fact familiar to all, that violent friction of bodies against each other will set combustibles on fire. The axles of railroad cars are made red-hot if not duly oiled; boats are set on fire by the rope drawn swiftly over the edge by a whale after he is harpooned; a stream of sparks flies from the emery wheel when steel is polished, etc.

A lecture illustration devised by Tyndall will show the conversion of motion into heat.

Screw a brass tube A, about 4 inches long and $\frac{1}{2}$ inch in diameter, upon the spindle of a whirling table B (Fig. 313). Nearly

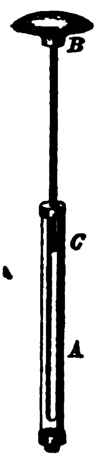
fill the tube with water and insert a cork; press the tube between the jaws of a wooden clamp *C*, while *A* is rapidly rotating. Heat is developed by the friction, and this communicated to the water causes it to boil, and finally to eject the cork.

FIG. 313.



The heat developed by sudden compression of air may be rendered visible by igniting vapor of carbon bisulphide in a "Fire Syringe." A thick glass tube *A* (Fig. 314) closed at the lower end, has a well-fitted piston *C*, whose rod is terminated by a wide cap, or button, *B*, upon which the palm of the hand may strike forcibly without injury. If a bit of tinder, or a small tuft of cotton moistened with carbon bisulphide, be placed at the bottom of the tube, it will be ignited when the piston is driven down by a sudden blow upon it.

FIG. 314.



Chemical action is another very common source of heat. Combustion is the effect of violent chemical attraction between atoms of different natures, when both light and heat are manifested. If the union goes on slowly, as in the rusting of iron, the amount of heat is the same, but it is diffused as fast as developed. The molecular energy, in most cases of chemical combination, as measured by the heating effect, is enormously great.

The warmth produced by the vital processes in plants and animals is caused by chemical action. In breathing the air, some of its oxygen is consumed, which becomes united with the blood. This process is in some respects analogous to a slow combustion, by which heat is evolved in the animal system.

PART VII.

ELECTRICITY AND MAGNETISM.

CHAPTER I.

ELECTROSTATICS.—POTENTIAL.—CAPACITY.

557. Definition.—The name *Electricity*, from the Greek word for *amber*, is given to a peculiar agency, which causes mutual attractions or repulsions between light bodies, and which, under proper conditions, also produces heat, light, sound, and chemical decomposition.

Lightning and thunder are familiar illustrations of the intense action of this agency.

558. Common Indications of Electricity.—If amber, sealing-wax, or any other resinous substance, be rubbed with dry woollen cloth, fur, or silk, and then brought near the face, the excited electricity disturbs the downy hairs upon the skin, and thus causes a sensation like that produced by a cobweb. When vulcanite is strongly excited, it gives off a spark to the finger held toward it, accompanied by a sharp snapping noise. A sheet of writing-paper, first dried by the fire, and then laid on a table and rubbed with India-rubber, becomes so much excited as to adhere to the wall of the room or any other surface to which it is applied. As the paper is pulled up slowly from the table by one edge, a number of small sparks may be seen and heard on the under side of the paper. In dry weather, the brushing of a garment causes the floating dust to fly back and cling to it.

Bodies are said to be electrically *excited* when they show signs of electricity in consequence of some mechanical action performed upon them, as in the experiments already described.

A body is *electrified* when it receives electricity, by communication, from another body already *excited* or *electrified*.

559. Repulsion.—An electrically excited body does not always produce attraction. It will be noticed that pith-balls, after

coming in contact with an electrified body, which has attracted them, are repelled. They have received a portion of the electricity which attracted them and repulsion is the result. This repulsion can be made much more apparent if an electrified vulcanite rod be suspended in a wire loop at the end of a silk thread and then a similarly electrified rod be approached to it. The suspended rod can be made to revolve rapidly because of the repulsion.

560. Theories of Electricity.—As to the exact nature of electricity science is still in the dark, though probably the darkness which precedes dawn.

Symmer proposed a "*two-fluid*" *theory*. He supposed every unelectrified body to contain equal quantities of two opposite kinds of imponderable electric fluid. In equal quantities they neutralized each other. But if, by friction or other means, the amount of one fluid be made to exceed that of the other, then the body becomes positively or negatively electrified. According to this theory two positively or two negatively electrified bodies repel each other; a positively electrified body and a negatively electrified body attract each other.

Franklin modified this into a "*one-fluid*" *theory*. Every body contains its own normal amount of one electric fluid. This amount is increased or decreased when rubbed by another body. The surplus amount is obtained from or given up to this second body. The body with more than its normal amount is positively electrified, and negatively electrified when it has less than this amount.

Lodge maintains that electricity is the luminiferous ether itself. He arrives at this conclusion after considering a great number of electrical phenomena which demand the ether for their proper explanation.

Without adopting any theory, electrical laws and phenomena may be understood by considering the fact that a body may be subject to two opposite electrical conditions. It may be positively or negatively electrified. The law regarding attraction and repulsion then is:

Similarly electrified bodies repel each other, and dissimilarly electrified bodies attract each other.

561. Electric Series.—If two bodies are rubbed together, one of them is electrified positively and the other negatively. One of these bodies, if rubbed by a third, may be oppositely electrified to what it was in the first case. Silk, when rubbed with glass, is negatively electrified; but rubbed with sulphur, it receives a positive charge. In the following series each member becomes

positively charged when rubbed on one following it, negatively when rubbed on one preceding it: *fur, wool, resin, glass, cotton, silk, wood, metals, sulphur, india-rubber, gutta-percha.*

562. Conductors and Insulators.—When a glass or vulcanite tube is rubbed with cat's fur, it shows that it has become electrified by attracting light articles. If a metal rod be substituted for the glass one, no attraction will be evidenced. This is not because the metal was not electrified by the rubbing, but because the electricity, as soon as generated, escaped, through the rod itself and the hand holding it, to the ground. If the rod be held by a glass or hard-rubber handle and then rubbed, it will attract as the glass did. This shows that some substances, as metals, allow electricity to pass freely through them, while others, as glass, almost entirely prevent its passage. The first class of substances are called *conductors*, the latter class *non-conductors* or *insulators*. Some substances neither conduct nor insulate well, but lie between the two classes. The following is a table of substances arranged in the order of their electrical conductivity:

CONDUCTORS.	4. Acids.	8. Wood.	12. Glass.
1. Metals.	5. Sea-water.	9. Silk.	13. Shellac.
2. Charcoal.	6. Vegetables.	10. India-rubber.	14. Vulcanite.
3. Graphite.	7. Animals.	11. Porcelain.	INSULATORS.

A conductor mounted upon or suspended by an insulator is said to be insulated.

A method for determining the conductivity of substances is to suspend two pith-balls by moistened threads from a metal insulated hook. Upon communicating a charge of electricity to the balls they will stand out away from each other, owing to the repulsion between the same kinds of electricity on each. If, now, one end of the substance, whose conductivity is to be determined, be held in the hand and the other be touched to the hook from which the balls are suspended, the rapidity with which the balls fall toward each other determines the conductivity. If they fall instantly, the substance is a good conductor. If they remain separated, the substance is a good insulator. After an insulator or an insulated conductor has been charged with electricity, the electricity of necessity remains at rest, and is, for this reason, called *statical electricity*. If, now, it be connected, by means of a conducting wire, with the moist earth, it will pass off instantly to the earth. During the time of its passage it is called *dynamical electricity*. If by chemical or other means the flow be maintained, then the dynamical electricity is called *galvanic* or *voltaic*.

563. Coulomb's Law.—Coulomb showed that, correspond-

ing to Newton's law of gravitation, *the force of attraction between dissimilarly electrified bodies and the force of repulsion between similarly electrified bodies is directly proportional to the product of the quantities of electricity and inversely proportional to the square of the distance between the bodies.*

If we represent the force by f dynes, the distance by r centimetres, and the quantities of electricity by q and q', then we can indicate the law by the equation

$$f = \frac{q\,q'}{r^2}.$$

If these magnitudes be connected by the sign of equality, a proper unit of quantity must be had. Letting $f = 1$ dyne, $r = 1$ centimeter, and $q = q'$, then $q^2 = 1$ and $q = \pm 1$. Hence we may define the unit of electrical quantity as follows:

One unit of electricity is that quantity which, when placed at a distance of one centimetre from a similar and equal quantity, repels it with a force of one dyne.

If the quantity of electricity be spread over a body of some size, as a sphere, then the distance r must be measured from some point as the centre of the sphere. This is evidently for the same reason as in gravitation, where the distance is measured from the centre of gravity.

It must be borne in mind that the unit of quantity here given is based upon the force exerted by two *statical* quantities of electricity. Another unit, based upon the *electro-magnetic* force, will be mentioned later.

564. Potential.—Whenever a body is lifted vertically away from the earth, the work performed in lifting it has been transformed into potential energy. The body has, because of the attraction between it and the earth, a potential energy capable of doing exactly the same number of ergs or foot-pounds of work as were used in raising it to its position (Art. 36). Similarly, if two conductors, charged with the same kind of electricity, be approached towards each other, a certain number of ergs of work will have been performed, owing to the repulsion between them. (A more perfect analogy would be to suppose two dissimilarly charged conductors to be separated.) The work which has been performed is also changed into potential energy between the conductors. The amount of energy made potential depends upon the quantities of electricity on each of the conductors, and upon the distance through which they have been moved toward each other. For energy is measured by the work it can do, and work in ergs equals the product of the force in dynes by the distance in centi-

metres through which it has acted. Now the force of repulsion between the two conductors equals the product of their quantities divided by the square of their distance apart.

Suppose one of the conductors to have any charge and to be fixed immovably. Then let three charges of respectively 1, 2, and 3 units of quantity be successively approached, between the same limits, towards the first conductor. In the first case a certain amount of energy will have been made potential; in the second case twice as much, and in the third three times as much. Evidently a certain amount of the energy made potential is owing to the immovable charge, and this amount is the same in each case. The condition of the space around an electrified body is termed the *potential*, owing to that charge. To obtain a quantitative expression for it, the movable charge must be taken of unit quantity. It must also be considered that the work necessary to approach an unit through a given distance is not as great as to approach it through twice that distance. Considering these two points, we have the definition of electrostatic potential:

The potential at any point is the work that must be spent upon a unit of positive electricity in bringing it up to that point from an infinite distance.

If the immovable charge be negative, no work would be required to move up a positive unit; on the contrary, work would be performed by the unit in travelling. Hence the potential, owing to a negative charge, is *negative potential*. It is convenient to consider it so.

565. Equipotential Surfaces.—If the charge of electricity be supposed to lie on a small sphere, then some point can be found on every possible radius of the sphere produced where the potential will be the same. That is, it would require the same amount of work to bring a positive unit of electricity from an infinite distance out on each radius to this point. In the case of a sphere being charged, these points would be equally distanced from the centre of the sphere. If now these points be connected together, a spherical surface will result. Any such surface which contains only points of the same potential is called an *equipotential surface.*

In order that an equipotential surface may be spherical, the charge must lie upon a sphere and must be free from other electrified bodies. If the electrified body be irregular in shape, the equipotential surfaces will be correspondingly so.

To transfer a quantity of electricity from one point in an equipotential surface to another in the same surface requires no work to

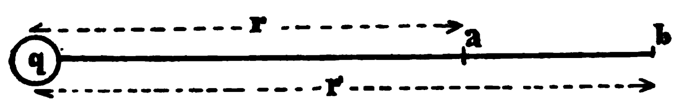
be performed. For while it may require work to move the charge in one direction from the surface, it will require a negative expenditure of work to bring it back again, *i.e.*, the attraction or repulsion between the electricities performs the work.

566. Difference of Potential.—In the consideration of most problems in electricity involving the idea of potential, the potential of two points is required. However, it is not the absolute potential of each of the points, but the difference of potential between them which is considered. If it requires a certain number of ergs to bring a unit of positive electricity from an infinite distance up to a given point, and more ergs to bring it up to a second point, then this extra work is what would be required to move the unit from the first to the second point. This number of ergs is then the measure of the difference of potential between the two points. Hence we obtain the definition:

The unit difference of potential is that which must exist between two points, that one erg may be required to move a positive unit of electricity from one to the other.

567. Unit of Potential.—The difference of potential between two points, a and b, Fig. 315, at distances r and r' from a quantity

FIG. 315.



of electricity q, is measured by the work necessary to move a positive unit of electricity from b to a.

This

work = *(average) force* × *distance through which it is overcome.*

The distance $= r' - r$.

$$\left.\begin{array}{l}\text{Force at } a = \dfrac{q}{r^2} \\ \text{Force at } b = \dfrac{q}{r'^2}\end{array}\right\} \text{average force} = \sqrt{\frac{q^2}{r^2\ r'^2}} = \frac{q^*}{r\ r'}$$

Hence the difference of potential

$$V_a - V_b = \frac{q}{r\ r'}(r' - r) = q\left(\frac{1}{r} - \frac{1}{r'}\right).$$

This equation for the difference of potential between two points enables us to obtain an equation for the absolute potential V_a at

* That this is a true average can be proved by a simple application of the calculus.

any point a. We have only to suppose that the second point b is removed to an infinite distance, where its potential $V_b = 0$, and $r' = \infty$. Hence

$$V_a = \frac{q}{r},$$

Or, in general,

The potential, V, of any point at a distance, r, from a quantity of electricity, q, is expressed by the equation,

$$V = \frac{q}{r}.$$

From this equation, supposing q and r each equal to unity, we obtain the definition:

The unit potential is that due to a unit quantity of electricity at a distance of one centimetre.

The potential at a point owing to several charges of electricity is equal to the sum of the potentials at that point due to each charge taken separately. Thus, if quantities of electricity q, q', and q'' a·e at distances r, r', and r'' from a point, the potential at that point

$$V = \frac{q}{r} + \frac{q'}{r'} + \frac{q''}{r''}.$$

568. Zero Potential.—At an infinite distance away from any electrified body the potential would evidently be zero. If a positive charge were brought near, the potential would become positive, and negative for a negative charge. In practice it is convenient to take the earth as a standard zero, with which all other potentials may be compared. This assumption is analogous to the use of the sea-level as the zero in measuring the heights of mountains instead of the centre of the earth.

569. Potential on a Sphere.—By discussing the equations in Art. 567, and supposing r equal to zero, one might be led to think that the potential would be infinite. But it must be remembered that, just as in gravitation, there is a centre from which all electric force apparently works. In the case of the earth all attraction is toward the centre of gravity. With an electrified sphere all action comes from the centre of the sphere. The electricity (Art. 572) lies upon the surface of it, but the resultant of all attractions from all the particles of electricity passes through the centre. Thus a point in the electricity itself on a charged conductor has a finite potential, and, *in the case of a sphere, it is equal to the quantity of electricity divided by the radius of the sphere.*

570. Capacity.—Suppose that a point on the surface of a charged spherical conductor, *i.e.*, any point in the electricity itself,

to have a certain potential. If, now, the radius of the sphere be supposed to grow smaller, while the quantity of electricity remains the same, then the potential will evidently increase as the radius decreases, because the potential $V = \frac{Q}{r}$. Thus a large sphere, *e.g.*, the earth, can hold a large quantity of electricity without having a high potential. This ratio between quantity and potential of electricity in a conductor is termed the *electrostatic capacity* of the conductor. Representing this by C, we have the definition in the form of an equation:

$$C = \frac{Q}{V}.$$

From this, by supposing Q and V each equal to unity, we have the definition,

That conductor has a unit of electrostatic capacity, which requires a unit quantity of electricity to raise its potential from zero to one.

Applying the above equation to a sphere of radius r we have

$$V = \frac{Q}{C} = \frac{Q}{r},$$

whence we see that the electrostatic capacities of spheres are equal to their radii. Accordingly a sphere of 1 centimetre radius has a unit capacity.

A conductor, no matter what its shape, will have a capacity, and we may say that,

The capacity of any conductor is equal to the number of units of quantity of electricity necessary to raise its potential from zero to unity.

571. Equipotential of Connected Conductors.—When a conductor is charged with electricity each particle strives to get out of the reach of its neighbors, because of the natural repulsion between like kinds of electricity. The particle, however, cannot escape, because the dry air is an insulator. If, now, it be connected, by means of a conducting wire, with the earth, the particle, followed by others, will flow off to the earth. Now the earth, having such a very large radius, would require an enormous quantity of electricity to raise its potential even an infinitesimal amount. The result is that the potential of the conductor and earth are both reduced to zero. Suppose, however, that instead of being connected with the earth it had been connected to another insulated conductor. The particles escaping from the first conductor would gradually raise the potential of the second until a particle, at some place on the connecting wire, would be equally repelled by the charges on each conductor, and would accordingly

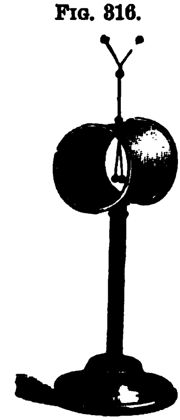
remain at rest. Now it will be found that, just as when two vessels, one of which contains water, when connected by a tube at the bottom, will allow the flow of water until the level in both is the same, so with these conductors, the potentials of both will have become the same because of the connecting wire. Furthermore, just as is the case with the connected vessels of water, it makes no difference whether the second conductor had originally a charge of electricity or not. The potential of all electrostatically charged conductors becomes the same when connected together.

If the potential of connected conductors becomes the same, then it is quite evident the total quantity of electricity must be so divided that each conductor shall have a quantity in direct proportion to its capacity. This must necessarily follow from the definition of capacity at the end of Art. 570. Thus three connected conductors of capacities 1, 2, and 3 would have respectively one-, two-, and three-sixths of the total quantity of electricity upon them.

572. Position of Static Charge.—A statical charge of electricity always resides on the outside surface of a conductor. It also resides on the outside of the geometrical figure of the conductor. Thus, if a charge be communicated to a wire bird-cage, it will reside wholly on the outside half of the wires and none will lie on the inside. This may be shown in many ways.

If a hollow, insulated, conducting cylinder (Fig. 316) be provided with two suspended pith balls in the interior and two on the exterior, and a charge of electricity be communicated to it, the outside balls will diverge, owing to the repulsion of like electricities. The inside balls will, on the other hand, remain at rest. It makes no difference whether the charge be communicated to the inside or outside. As soon as it has been communicated the inside balls drop to their normal position.

FIG. 316.



In calculating the capacities of conductors, it makes no difference whether a conductor is solid or hollow.

573. Distribution of a Charge on the Surface.—Statical electricity resides at the surface of a body, as we have seen, but is not uniformly diffused over it, except in the case of the sphere. In general, the more *prominent* the part, and the more *rapid its curvature*, the more intensely is the electricity accumulated there.

In a long, slender rod the density is greatest at the ends, nearly the whole charge being collected at these points. On a sphere,

not influenced by other electrified bodies, the density is uniform, as illustrated in Fig. 317, the dotted line denoting by its constant distance from the surface the uniform distribution of the charge.

Fig. 318 represents the varying density upon an ellipsoid.

FIG. 317.

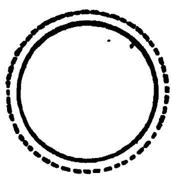
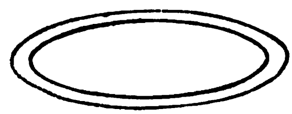


FIG. 318.



The two ellipsoids are similar, and the ellipsoidal shell included between them represents the densities at various points. In this case the densities at any two points of the ellipsoid are nearly proportional to the diameters through those points.

The student must remember that the charge does not form a layer upon the body in any sense whatever, and that the above figures are given merely to aid the memory in retaining the law of distribution.

574. Surface Density.—The greater the quantity of electricity on a given conductor, the greater the tendency is for the electricity to escape to surrounding objects.

The surface density at any point of a surface, when the distribution is uniform, is the quantity of electricity per square centimetre of surface.

If Q units of electricity reside on S square centimetres of surface, then the surface density d is represented by the formula

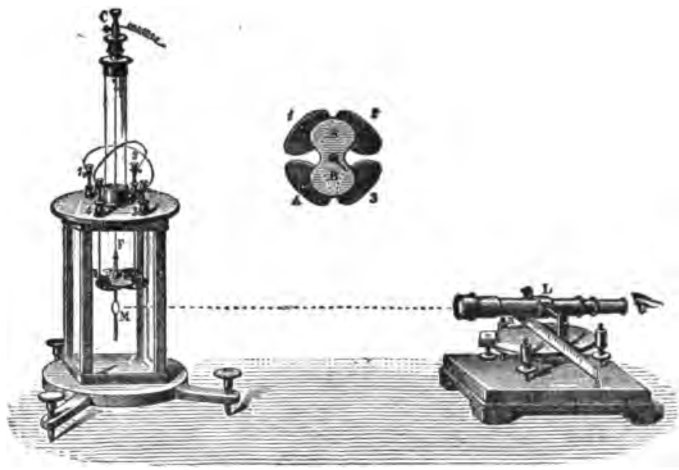
$$d = \frac{Q}{S}.$$

The surface of a fine point is very small, hence, if there is any quantity of electricity supplied to it, the density becomes very great and the charge escapes into the air.

575. Quadrant Electrometer.—This instrument is used for determining very accurately differences of electrical potential. A simple form, suited for qualitative work, is shown in Fig. 319. Four like pieces of metal are suspended, by conducting rods, from the insulating top of a glass case. They are symmetrically placed (as shown in the small figure) and are fixed in position. Over these quadrants, as they are termed, swings a flat aluminum needle, suspended by a wire of small diameter. This prolonged suspension hangs in a glass chimney, placed upon the top of the

case. A small mirror, M, is attached to a rigid prolongation of the suspension, prolonged beneath the needle. This mirror

FIG. 319.



serves to reflect the image of a scale, S, through a reading telescope, L, by means of which deflections of the needle can be observed.

To use the instrument, the needle is charged, through its suspending wire, to a constant potential. This may be done by connecting C with the knob of a charged Leyden jar. The diagonally opposite quadrants are connected together, and the two pairs connected with the points whose difference of potential is to be determined. Now suppose that 2 and 4 (small diagram) were of higher potential than 1 and 3. They would exert a greater force upon the needle than 1 and 3, and according to the sign of the charge on the needle, would cause rotation of the needle in one direction or another. The needle, which was held in the zero position by the torsion of its suspension, would come to rest at a place where the force of torsion was equal and opposed to the electrical force. For small deflections the forces are proportional to the tangents of the angle, *i.e.*, to the readings of the scale.

For very accurate work many complicated attachments are added to this simple form.

Problems.

1. Two conductors, of capacity 10 and 15 respectively, are connected by a fine wire and a charge of 1000 units is divided between them: find the charge which each takes, and the potential to which each is raised.

2. Three spheres of radii 1, 2, and 3 cm. are charged to potentials 3, 2, and 1 respectively, and are then connected by a fine wire: what is their common potential?

3. Two spheres, of capacity 2 and 3, are charged respectively to potentials 5 and 10: what will be their common potential, if they are placed in electrical connection?

4. Two spheres, of 2 and 6 cm. radius, are charged respectively with 80 and 30 units of electricity; compare their potentials. If they are connected by a fine wire, how much electricity will pass along it?

5. Twelve units of electricity raises the potential of a conductor from 0 to 3: what is its capacity?

CHAPTER II.

ELECTROSTATIC INDUCTION.

576. Gold-leaf Electroscope.—The gold-leaf electroscope is a delicate instrument for detecting the presence of electricity. It consists (Fig. 320) of a folded strip of gold-leaf, suspended from the end of a brass rod, which penetrates the stopper of a glass insulating receiver. The outside end of the rod is provided with a brass ball. Whenever a charge of electricity is communicated to the ball the gold-leaves partake of it and diverge from each other, because of the repulsion of like kinds of electricity. The sides of the receiver are provided with strips of tin-foil, which are in electrical communication with the earth through the base. The object of these is to prevent the rupturing of the gold-leaves by the sudden communication of too great a charge. Upon receiving such a charge they diverge and communicate it to the tin-foil and it escapes thence to the earth.

FIG. 320.

577. Phenomena of Induction.—Whenever an electrified body is approached toward the brass ball of an electroscope, it will be noticed that, while it is even a great distance away from it, the gold-leaves begin to separate and show the presence of electricity upon them. This electricity is the result of the presence

of an electrified body in the neighborhood and is called *induced* electricity. The process under which it was generated is termed *electrostatic induction.*

Whenever a charged conductor is brought near to an uncharged conductor, and is separated from it only by an insulator, which in this case is called a *dielectric*, the uncharged conductor undergoes an electrical change. The side which is toward the first conductor is charged with an opposite kind of electricity, while the remote side has a charge of same kind as the original charge.

Thus (Fig. 320), if a negatively electrified piece of hard rubber be brought near to the gold-leaf electroscope and is separated from it by air for a dielectric, there will be positive electricity induced on the nearer side of the electroscope, which is the ball, and negative on the remote, which includes the gold-leaves. The leaves accordingly diverge. The electricity on A is called the *inducing* charge, that in the electroscope the *induced charge.*

Whenever an insulated conductor, which contains the two kinds of induced electricity and is still under the influence of the inducing charge, is connected with the earth, the electricity of the same kind as the inducing charge will escape to the earth. This is because of the repulsion between like kinds of electricity. It is equally true whether the near or remote side of the conductor is connected to the earth.

The remaining opposite kind, however, cannot escape because of the attraction exerted by the original inducing charge. If now the earth connection be removed, it will be found that only a small portion of the original inducing charge can escape when connected to the earth. It is held in place by an opposite kind of electricity, which it has itself produced. These two opposite electricities, separated by a dielectric, are said to be *bound*, while electricity free to follow an earth connection is called *free* electricity.

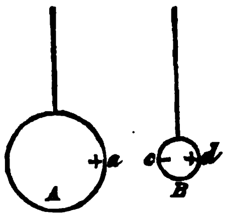
For illustration, suppose that the apparatus is in the condition represented in Fig. 320. If the finger be touched at C, the electricity $n\ n$ will escape to the earth and the leaves will collapse. The positive charge at C remains bound by A's charge. If now A be removed, this charge will diffuse over the electroscope and the leaves will diverge because of the portions which they receive.

As might be expected, successive inductions may be obtained from one original charge. The induced charge in one case acts as the inducing charge in a new induction.

578. Induction Precedes Attraction.—Whenever a body is attracted because of the charge of electricity on another body, it is always subjected to induction before it is attracted.

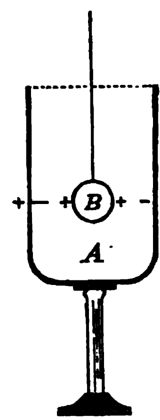
Thus, if B (Fig. 321) is attracted by a positive charge on A, the attraction is always preceded by an induction, whereby B is charged negatively at c and positively at d; c is nearer than d, hence the attraction between a and c is greater than the repulsion between a and d. Accordingly attraction predominates.

FIG. 321.



579. Quantity of the Induced Electricity.—The *total* quantity of electricity, of the opposite kind to its own, which a charged body induces on neighboring bodies is exactly equal to its own charge. This was experimentally proved by Faraday by means of an "ice-pail." A metallic pail, A (Fig. 322), was mounted upon an insulating support. The outside of the pail was connected with a delicate electroscope. Into this pail was lowered a positively charged ball, B, which was suspended from an insulating silk thread. Upon introducing the ball the leaves of the electroscope commenced to diverge, because the charge on B induced negative electricity on the interior of the pail and held it bound. The positive electricity of the pail, being free and repelled, passed partly into the electroscope. As the ball was lowered further the leaves diverged more and more until, after a certain depth had been reached, a further descent produced no extra divergence. Even when the ball was brought into contact with the bottom of the pail the leaves remained undisturbed and extended. Upon removing the ball, after contact, the charge was found to have disappeared from it. The fact that the gold-leaves were undisturbed by the contact of the ball with the pail proves that there was the same quantity of negative electricity on the inside of the pail as positive electricity on the ball. Coming together the two neutralized each other and left the positive outside charge undisturbed.

FIG. 322.



580. Condensers.—If a pane of glass be taken, and a piece of tin-foil be pasted upon the middle of each face of the pane, and one piece be charged positively, the inner surface of the other piece will receive a negative charge by induction. If the second piece be connected with the earth positive electricity will escape. The positive electricity of the first tin-foil will attract and hold the negative of the second *bound*. If the connections to the source

and the earth be now removed, it will be found that hardly any electricity can be obtained by merely touching either of the foils. It may be said that each charge is inducing the other. It will be found that these two pieces of tin-foil may be, when thus arranged, much more strongly charged than either of them could possibly be, if it were placed alone upon a piece of glass and then electrified. In other words, the capacity of a conductor is greatly increased when it is placed near to a conductor electrified with the opposite kind of charge. Considering then (Art. 570) that the potential $V = \frac{Q}{C}$, it will be seen that such a piece of apparatus can receive a large quantity of electricity, Q, without raising its potential, V, as much as if it were separated from all conducting or electrified bodies. Such an arrangement is called a *condenser.*

Condensers are much used in practical electricity for measuring quantities of electricity. A pane of glass, however, would not have a sufficiently large capacity for technical purposes. Accordingly commercial condensers are made by piling together alternate sheets of tin-foil and paraffined paper. The alternate layers of tin-foil are connected together. In this manner a large surface of tin-foil can be used and yet not occupy an inordinate amount of space. The capacity of a condenser varies inversely as the thickness of the *dielectric* between the conducting sheets, and directly as the product of the area of the sheets and a constant depending upon the nature of the dielectric. This constant is called the *specific inductive capacity.*

581. Specific Inductive Capacity.—It has been stated (Art. 563) that a body charged with Q units of electricity will attract an unlike unit of electricity on a body which is r centimetres away with a force

$$F = \frac{Q}{r^2}.$$

This is strictly true only when the two bodies are in a vacuum. It is very nearly true when they are separated from each other by dry air or any other gas. That the expression for the force may be universally true, whatever be the dielectric which intervenes between the bodies, it must be modified into the form

$$F = \frac{Q}{K\,r^2}.$$

Here K is a constant, which is peculiar to each substance, and it is called the *specific inductive capacity* or the *dielectric constant* of the substance.

The following is a table of specific inductive capacities referred to air at 0° C. and 760 mm. pressure as unity:

Air and most gases	1.0
Bisulphide of Carbon	2.2
Ebonite and Rubber	2.3
Paraffin	2.3
Shellac	2.9
Sulphur	3.7
Glass	3.2 to 6.0
Water	6.0
Metals	∞

The name "inductive capacity" was introduced by Faraday in the publication of a series of experiments upon condensers. He constructed a number of exactly similar condensers, differing from each other only in the dielectric between the conducting surfaces. The dielectrics which he used were air, sulphur, shellac, and glass. He measured the capacities of these equally dimensioned condensers and found that all the others had greater capacities than the one containing air. Remembering that induction is the principle upon which the condenser works, it can readily be seen why Faraday adopted the term.

Modern writers, however, in employing the term "dielectric constant" indicate their appreciation of the important light thrown, by the mere existence of different values of K, upon the true nature of electricity. The fact that the nature of the substance between two charges of electricity influences the magnitude of their repulsions, disproves the idea that electrical attractions and repulsions are *action at a distance.* There must be something between the bodies which plays a part. Again, the repulsion of two charges of electricity, even when suspended in a vacuum, indicates that this something must be the ether. The ether, then, in different dielectrics, must in some manner be modified from what it is in a vacuum. This is known to be the case from the different optical properties of bodies.

A method for showing directly the effect of K in Coulomb's law has been constructed by Mayer. Suspend horizontally a silvered circular disc of mica, 16 cms. in diameter, by a long, slender, spiral spring. Let the spring be in contact with the silvered surface. Under this disc place another of metal, which is movable in a vertical direction, and is connected to the earth. Charge the silvered mica with electricity, using the spring as a means of connection. If the distance between the two discs is but two or three centimetres, the mica will be attracted so as to extend the spring. If, now, a sheet of paraffin or plate of glass be interposed between the two discs, the attractive force will be observed to

decrease. This is because K in the denominator of the fraction expressing the force of attraction is greater for glass and paraffin than for air.

It will be noticed, in the table given, that K for metals and conductors is ∞. According to this, a charged body cannot attract a pith-ball through a metal plate. The metal is therefore called an electric *screen*. The screen must be sufficiently large to prevent any attractive force from working around its edges, and it should be connected with the earth to avoid any induced electricity from exerting an influence. It is in consideration of these facts that electrometers are surrounded by conducting cages, which are connected with the earth. Any neighboring accidental charges of electricity cannot then influence the electrometer needle.

582. **Leyden Jar.**—A most convenient form of condenser, for demonstrative experiments, is the Leyden jar. It usually consists (Fig. 323) of a glass jar coated up to a certain height on the inside and outside with tin-foil. A brass knob fixed on the end of a stout brass wire passes downward through an insulating lid and is connected by a chain with the interior coating of tin-foil. To charge the jar, it is held, by the outer coating, in the hand, and the brass knob is approached to any source of electricity. If the source furnishes positive then the internal coating becomes charged positively, and this induces and binds an equal amount of negative electricity on the outer coating, while an equal amount of positive will be rendered free and will escape through the hand to the ground. The jar being now removed, is said to be charged—there exists a state of positive potential on the inside and negative potential on the outside. Both electricities are bound, and neither can produce effects independently. If, however, they be allowed to come in connection with each other (by joining the outside coating and the ball at the top with a wire) the electricities rush to neutralize each other, and will even spark across an air gap. The jar is then said to be *discharged*. The length of the air gap through which the spark will pass depends upon the difference of potential between the inner and outer coatings. Sometimes the difference of potential becomes so great, owing to carelessness in charging, that the electricities, in striving to get together, will pierce and fracture the glass itself.

Fig. 323.

583. **Seat of the Charge.**—If a jar is made with a wide, open top, and the coatings movable, then, after charging the jar, the coatings may be removed and tested without showing any trace of

electricity upon them. If they be then replaced, the jar will be found to be charged as before removal. Benjamin Franklin inferred from this that the electricity resides upon the dielectric and the coatings serve only to readily diffuse the charge over the surface.

584. Residual Charge.—If a jar be charged, left for a time, discharged, and left for a while longer, it will be found that, upon connecting the two coatings, a spark may be obtained. The electricity remaining in the jar after the first discharge is called the *residual charge*. The amount of residual charge varies with the time that the jar has been left charged. It also depends upon the kind of dielectric used. No residual charge has been found in connection with an air-condenser.

585. Modern Theory of Condensers.—The modern ether theory of electricity gives a very satisfactory explanation of condensers. According to this theory electricity is the ether itself. When a conductor is statically charged, it is not the ether of the conductor which constitutes the charge, but the ether of the dielectric which surrounds the conductor. More exactly, charging a conductor is straining the ether particles of the surrounding dielectric out of a definite position, which they are presumed to have a tendency to remain in. Thus, if a positive charge is communicated to the inner coating of a Leyden jar, the ether particles of the glass will all be strained away from the inside. All the particles will be strained away from the inner coating and toward the outer coating. We may thus say that the inner is positively electrified and the outer negatively. We have but one ether electrification, but two ways of looking at it—just as a dent in a tin plate may be convex on one side and concave on the other.

In all dielectrics the ether particles are supposed to be held in position by elastic ties of some sort. A mechanical analogy is to represent the particle (Fig. 324) by a bead fastened to the centre of an elastic wire, clamped at both sides. When subjected to an electrifying influence, the bead is pulled to one side. Upon releasing the bead, or discharging the electricity, two things are to be noticed. First, the bead will not only go back to its original position, but will go beyond it and oscillate a number of times before coming to rest. According to this, then, the spark, at discharging a Leyden jar, should be oscillatory and made up of a succession of small sparks. Such is the case, as has been shown by reflection upon a screen from a rapidly rotating mirror. Were the spark a continuous one, its reflected

FIG. 324.

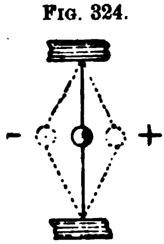


image would appear as a prolonged line. It appears, however, as a dotted line. Secondly, it must be considered that, if the strain on the wire be maintained for any length of time, it will not, upon release, immediately return to its normal position, but will assume a new one, which is displaced in the direction of the strain from the original position. This is a common phenomenon and is known as *elastic fatigue.* The electrical parallel is the residual charge. After the first discharge the ether particles have not returned to their normal position. It requires one or more residual discharges before they return to that position.

Of course in a dielectric we cannot suppose that each of the infinite number of particles of ether is supported upon anything similar to an elastic wire, for the dielectric would act well in one direction and not at all in the direction of the length of the wire. This is not in accordance with facts. Accordingly it is supposed that the dielectric acts as a mass of jelly, in which its ether particles are suspended, or possibly the ether is a jelly-like mass in itself, lacking, however, the physical property impenetrability. The jelly then exerts a restraining force to strains exerted in *any* direction upon the particles.

Conductors, on the other hand, are supposed to exercise little or no restraint upon the free movement of the ether within them.

Viewed in this light, when a positive charge is communicated to a conductor, an attempt is made to force more ether into the conductor than it ordinarily has. But the ether is absolutely incompressible. Hence room is made for the extra amount, by pressing the ether of the dielectric away from the conductor. The extra ether must have been taken from somewhere, and the place which it formerly occupied must be filled. This is done by the distention of the remote portions of the dielectric. It presses out of some conductor an equal amount of ether. We can thus see the truth of Faraday's statement that it is impossible to charge one body alone. Whenever a body is charged positively, some other body or bodies must receive an equal negative charge.

When a Leyden jar is discharged, by connecting the two coatings through a conductor, the ether in the positive coating flows toward the negative and the strain on the dielectric is relieved. This flow will, of course, be vibratory, owing to the inertia of the dielectric jelly.

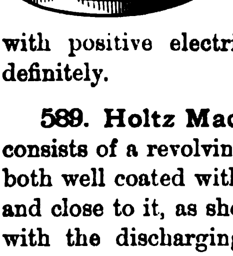
566. Hertz's Experiments.—Professor Hertz has shown experimentally that the electrical ether is wonderfully like, if not identical with, the ether presupposed in light. By rapidly charging and discharging a conductor he causes the ether upon it to surge

to and fro. This agitates the surrounding dielectric ether, and the disturbances travel in waves. The velocity of propagation he determines to be the same as the velocity of light. He has made these waves interfere, has reflected them from metal mirrors, refracted them through lenses and prisms of pitch, and has produced diffraction effects. He has shown that many optical experiments can be electrically performed by substituting dielectrics for transparent and conductors for opaque bodies.

587. Electrical Machines.—The method of rubbing sealing-wax with fur is too slow for the production of large quantities of electricity. Franklin improved upon this method, employing a machine, which rotated a large glass cylinder, the cylinder being rubbed by a silk rubber. But, at present, machines which generate electricity by friction are little used, recourse being made instead to the principle of induction. Two machines of this sort, most commonly found, are the electrophorus and the Holtz machine.

588. Electrophorus.—The electrophorus consists of a circular cake of resin, sulphur, or vulcanized rubber in a metallic base, *A* (Fig. 325), and a metallic disc, *B*, having an insulating handle. Stroking with flannel or fur electrifies the resin negatively. This induces and binds positive electricity on the lower face of the disc, when placed upon it. Free negative is repelled to the upper face. If the finger be touched to the disc, while it yet remains upon the resin, the free negative will escape to the earth. Upon then lifting the disc it will be found charged with positive electricity. This operation may be repeated indefinitely.

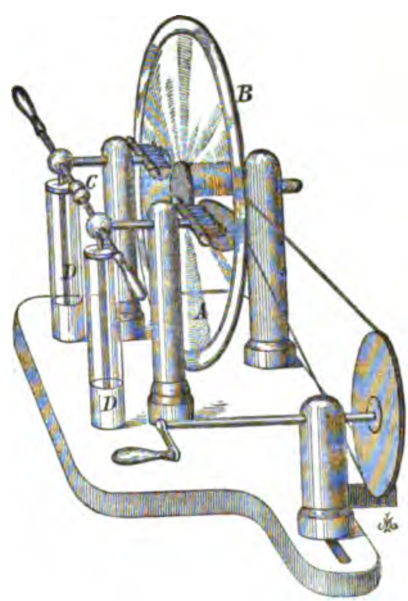
FIG. 325.



589. Holtz Machine.—This machine, illustrated in Fig. 326, consists of a revolving glass disc, *A*, and a stationary glass disc, *B*, both well coated with shellac to further insulation. In front of *A*, and close to it, as shown in the figure, are two combs connected with the discharging knobs at *C*. On the back of the disc *B*, opposite to the combs, are two paper sectors, a paper tongue or point from each projecting, through an opening (window) in the stationary disc, toward the revolving plate. If a plate of vulcanite be excited, and then be laid against one of the paper sectors, while the disc *A* is rapidly rotated *toward the point of the sectors*, the discharging knobs being in contact, electrical induction

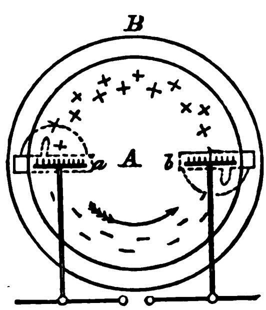
will ensue, and after a few moments the knobs may be *gradually* separated until sparks perhaps 12 or 20 inches long pass, according to the size of the machine. To produce sparks of great density two Leyden jars, *D*, *D*, with their inner coatings connected to the discharging knobs and their outer coatings connected with each other, are added.

FIG. 326.



To explain, in a very general way, the action of the machine, let *A* (Fig. 327) represent the revolving plate and *B* the stationary disc behind it, carrying the paper sectors *a* and *b*. Imagine the combs in front of *A*, and call them *a* and *b* also, remembering that the sectors are on the plate *B*, behind *A*. If now a positive charge be communicated to the *sector a*, it will act inductively upon the *comb a*, through the revolving plate, as a dielectric. Negative electricity will be induced in the *comb*, and, if it were of proper shape, would be bound. But, being pointed, the negative charge escapes to the surface of *A* and is carried around with it. The positive electricity of the comb is repelled by the process of induction to the discharging knob connected with it. Both knobs being in contact it passes to the *comb b*. From here it escapes to the front of the revolving disc, and at the same time induces negative electricity on the upper portion of the *sector b*. Because of the induction positive electricity escapes from the point of the *sector b* to the back of the revolving disc. If now the disc be revolved half-way around, the positive charge on the back of *A* will be taken up by the point of *sector a*, thus strengthening its original charge. *Sector b*, having now become charged negatively increases the flow of positive from

FIG. 327.



comb b, and *sector a*, having its charge increased, still further increases this flow. All the arrangements conspire to charge both the rear and front of the upper half of *A* with positive, and of the lower half with negative, electricity. The action requires that positive electricity shall flow continuously through the knobs from *a* to *b*. When sufficient potential difference between the combs has been obtained, the discharging knobs may be separated and a spark will rupture the air.

The Leyden jars serve to increase the capacity of the knobs, and thus, for a spark at a given potential difference, to increase the quantity of electricity passed.

A modification of the Holtz machine has been made by Wimshurst. The two plates, furnished with numerous sectors of tin-foil, are rotated in opposite directions. A full description is out of place here.

590. Effects of Statical Discharge.—Most electrical effects are best obtained by the use of current electricity. Those which require the high potential of statical electricity are the following:

MECHANICAL.—If a heavily charged Leyden jar be made to discharge itself through a piece of glass or card-board, it will, by the passage, pierce a hole through the piece. In case card-board is used, it will be found that there is a burr on both sides of the card. This is because of the vibratory character of the discharge.

PHYSIOLOGICAL.—If a discharge is made through a human being, the muscles which lie along its path will be strongly contracted. Those who have received very powerful shocks from electrical discharges say that the feeling is as though all the muscles had been so severely contracted as to result in spraining them. The action of the electricity is through the medium of the nerves.

HEATING.—A very sudden development of heat accompanies the spark at discharge. This can be easily shown by allowing a spark to pass to the tip of a gas-burner. If the gas be turned on, it will become ignited. If gas be not available, the spark may be passed to a spoon containing a few drops of common ether.

591. Lightning.—Water, in the process of evaporation, is supposed to become electrified. From the surface of large bodies of water multitudes of small electrified particles of moisture rise into the air, under the influence of the hot sun. These particles have a definite capacity (their radius) and a definite quantity of electricity. In the process of cloud formation these particles come into drops. Each drop receives the electricity of its component particles and has its capacity increased. The quantity on the drop equals the sum of the quantities on the particles. The

capacity of the drop is *less* than the sum of the capacities of the particles. Hence the potential on the drop is greater than it was upon the particles. In this manner clouds are formed, having large quantities of electricity at a high potential. The opposite kind of electricity is induced in the earth, and the air, acting as a dielectric, is placed under severe strain. Eventually the strain becomes too great and the air gives way. The equalization of potential, at that instant, gives what is termed a stroke of lightning. The intense heat developed by the discharge expands the air, and the rushing of cold air into the partial vacuum, thus formed, produces the sound thunder.

Sheet lightning, where a large surface is momentarily illuminated, is but the reflection from a cloud of an invisible true discharge.

CHAPTER III.

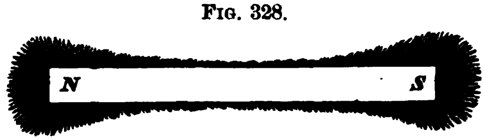
MAGNETISM.

592. Natural Magnets.—The ancients discovered that a certain black stone, abundantly found in Magnesia, had the property of attracting to it small pieces of iron. Accordingly, from their source, they called these stones magnets. Afterwards they found that, when hung by threads, a certain part of each stone always pointed north. From this property the stone received the name *Lodestone* (leading stone).

593. Artificial Magnets.—If a piece of steel be rubbed with a lodestone, it will be found to have acquired the property of attraction. Steel artificial magnets are what are employed in experiments in the laboratory.

594. Poles of a Magnet.—If a steel-bar magnet be rolled in iron filings (Fig. 328), it will be observed that the attractions seem to have two common sources, two points near the ends of the bar. These two points are called the *poles* of the magnet. The straight line connecting the poles is called the *magnetic axis*.

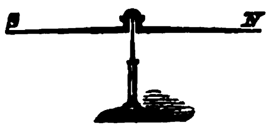
FIG. 328.



595. Magnetic Needle.—For investigating the attractions

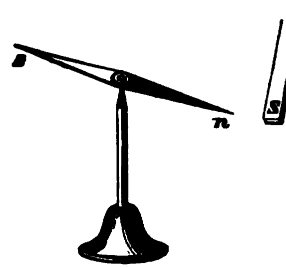
of magnets use is made of the magnetic needle. This consists (Fig. 329) of a light steel needle, which has been magnetized, and, by some suitable arrangement, is mounted upon a pivot. It is capable of moving in a horizontal plane with little friction. Left to itself it will assume a north and south direction. That end of it which points north is called the *north pole*, and the other end the *south pole*. The *compasses* sold by opticians are magnetic needles, whose north poles are generally more pointed than their south poles.

FIG. 329.



596. Attractions and Repulsions.—If a piece of iron be approached, in any manner, to either end of a magnetic needle, the needle will be attracted toward the iron. The same result will follow if either end of a magnet be approached to an iron non-magnetic needle. However, if either end of a magnet be approached to a magnetic needle (Fig. 330) attraction will follow when the adjacent poles are unlike, and repulsion takes place when the adjacent poles are of the same kind. Hence, as with quantities of electricity:

FIG. 330.



Poles of the same name repel, and those of contrary name attract each other.

597. North and South Poles Inseparable.—If any one portion of a piece of steel be touched by a north pole of a lodestone, it will be found to have developed a south pole. At the same time, however, a north pole has been developed in some other part of the steel. Again, if a bar magnet be broken at a point half-way between its poles, each of the fragments will possess two poles. Successive breaking leaves each fragment with its two different poles. The end of a fragment which had a pole before the rupture retains the same polarity afterwards. It may thus be concluded that every magnet must have two poles.

598. Magnetic Induction.—When a bar of iron is brought near to the pole of a magnet, though attraction is the phenomenon first observed, yet it is readily proved that this attraction results from a change, which is previously produced in the iron. Similar to the case of electrostatic attraction, the iron becomes a magnet

by *induction* exerted by the original magnet. By moving a magnetic needle around the iron it will be found that the end of the iron which is placed near one pole of the magnet becomes a pole of the opposite name, and the remote end a pole of the same name. Hence the adjacent, unlike poles of the iron and magnet, attract each other.

The induced magnet is more powerful the nearer it is to the inducing magnet; it is, therefore, greatest when the two bars are in contact.

Soft iron retains its magnetic properties only while under the influence of the magnet. Upon removing, it will be found to have returned to its neutral state. Had steel, or impure iron, or cast-iron been used, it would have been found to have retained more or less of the magnetic properties caused by the induction.

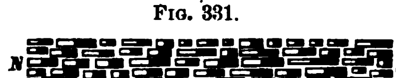
The inductive action may be well seen by placing iron and magnet upon a sheet of paper and then sifting iron-filings upon them. The filings will attach themselves to the iron in the same manner as to the magnet. If the magnet be now withdrawn, the filings around the iron will collapse, showing the loss of magnetic polarity.

The induced temporary magnet will, in its turn, induce temporary magnetism in a second piece of iron, and this again in a third piece. The strength, however, decreases as the pieces get farther away from the original magnet.

If the north ends of two equal magnets be touched to the opposite ends of a bar of steel, south poles will be induced in both ends of the bar. But we have seen that every south pole must have a north pole with it. Accordingly examination will reveal that the bar has two coincident north poles at its middle. Such intermediate poles are termed *consequent poles.*

599. Retentivity or Coercive Force.—The extension of the experiment of breaking a magnet (Art. 597) leads to the inference that every particle of steel is a magnet in itself. Before magnetization these molecular magnets point in all directions, and hence exert no external magnetic influence. Under the influence of induction, however, these are made to assume the same direction. Fig. 331 gives an idea of the probable arrangement of a magnetized piece of steel. The shaded ends represent the south poles of the molecular magnets. When they are all arranged as in the figure the external effect is as though there were a south pole at S and a north pole at N.

Fig. 331.



Now experiments show that tempered steel is much more difficult to magnetize than a piece of soft iron, and, after being once magnetized, retains its magnetism much better. It is, then, reasonable to suppose that the existence of foreign particles (carbon) in the steel hinders and clogs the turning of the molecular magnets from their chaotic state into regular arrangement. Once arranged, the same cause prevents a disarrangement. In pure iron the hindrance is not present. This resistance against a magnetizing or demagnetizing force is called *retentivity* or *coercive force*. As might be expected, the retentivity is modified by anything which will cause the molecules to vibrate, as hitting sharp blows with a hammer or heating to a high temperature. A magnet may be demagnetized by dropping it several times upon the floor.

The extreme amount of magnetism that could be imparted to a bar would be that which arranged all the molecular magnets in the same direction. The magnet is then said to be *saturated*. After removing the magnetizing force, however, some of the molecular magnets would of their own accord turn from line and others would follow their example in time. Hence a magnet must be kept for some time before its strength can be considered as constant. Yet a constant strength may be obtained by "cooking" the magnet. It is saturated and then placed for several hours in a bath of steam, removed and again saturated and cooked. Magnets treated in this manner are said to remain very constant.

600. Law of Magnetic Force.—Magnetic attractions and repulsions take place according to a law similar to Coulomb's law for electrical forces. Two like isolated magnetic poles of strengths m and m', d centimetres from each other, will repel each other with a force in dynes,

$$F = \frac{m\,m'}{d^2}.$$

If the poles were different, as m and $-m'$, then the value of F would be negative. A negative value indicates attraction, and a positive, repulsion.

The magnetic force will act through all substances except through magnetic substances, *i.e.*, those which are attracted by a magnet. No attraction can take place through a large iron sheet. Such a piece of iron is called a magnetic *screen*. A small magnet suspended in a hollow iron sphere cannot be deflected by an outside magnet.

601. Unit Magnet Pole.—If, in the formula for the force, given in the previous article, F and d be supposed each equal to unity, then, as in electrostatics (Art. 563),

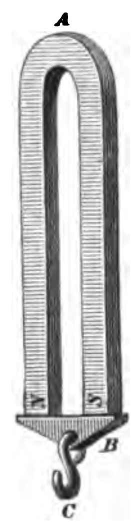
The unit magnetic pole is one that will repel an equal like pole, when at a unit's distance, with a unit force.

Of course an isolated pole cannot be obtained, for, in a magnet, it is always accompanied by an opposite equal pole, and the algebraic sum of the strengths of the poles of a magnet always equals zero.

The poles of a bar magnet are the points from which all the forces may be considered to emanate. If the strength of *one* of these poles be multiplied by the distance between the two poles, a quantity results which is termed the *magnetic moment* of the bar. In ordinary bar magnets the *pole distance* is about $\frac{5}{6}$ of the total length of the bar.

The magnetic moment of a bar divided by its weight in grams gives the *specific magnetism* of the substance of which the bar is composed. This is greatest in very hard-tempered steel. The magnetic moment divided by the volume of a magnet, *i.e.*, the magnet strength per unit volume, is termed the *intensity of magnetization* and is generally represented by the letter **I**.

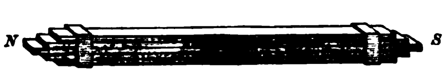
Fig. 332.



602. Lifting Power.—The strength of a magnet must not be confounded with its lifting power. The latter depends upon the shape of the magnet and also upon the shape of the body lifted. A magnet bent into the shape of a horseshoe (Fig. 332) will lift about four times what it would with one end, if straight. The lifting power of a magnet grows very curiously, if the load be gradually increased from time to time.

603. Laminated Magnets.—Long, thin steel magnets are more powerful in proportion to their weight than thicker ones. Hence compound magnets are constructed, consisting of thin laminæ of steel separately magnetized and afterwards bound together in bundles. These laminated magnets (Fig. 333) are more powerful than simple bars of steel. The explanation of this fact seems to be that ordinary steel magnets are never saturated, and what magnetism they have results from molecular arrangements near the surface. The compound magnets have a greater surface and are hence stronger. Since the mutual action of the like poles in juxtaposition tends to weaken them, the strength of a compound magnet will never equal the sum of the strengths of its parts.

Fig. 333.

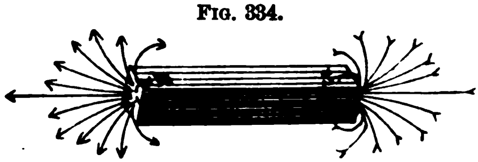


That the magnetism of ordinary bars is confined to the surface has been shown by placing the magnet in acid and dissolving its surface. After removal, the bar showed very little magnetic polarity.

604. Magnetic Field.—Lines of Force.—The space around a magnet where its action is felt is termed the *field* of the magnet. When several magnets are near to each other each one furnishes its own field, and superposed upon each other they form a resultant field.

The field is supposed to be permeated by magnetic *lines of force.* These lines represent the direction along which the magnetic attractions and repulsions act. An isolated magnetic pole would move along one of these lines under the attraction exerted by the field magnet. The general direction of the lines of force of a bar magnet are represented in Fig. 334.

FIG. 334.



The properties of these lines can be best discussed by considering only those which lie in a given plane passing through the magnet. Such a section is called a *magnetic spectrum.* The spectrum may be graphically represented by placing a sheet of white paper over a magnet and then sifting fine iron-filings upon the paper. A slight tapping on the paper will cause the filings to arrange themselves along the lines of force, as represented in

FIG. 335.

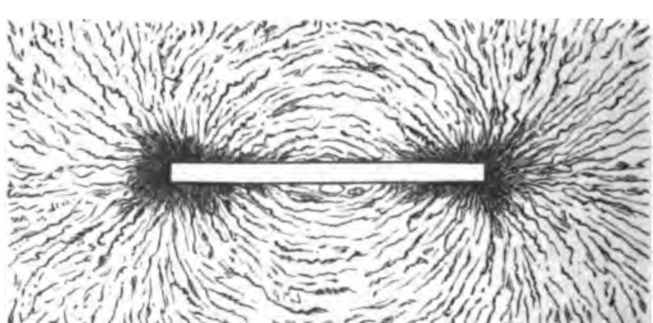


Fig. 335. With a sufficiently large figure it would be seen that every line starting from one pole finds its way, by a curved path, to the other pole.

An isolated north magnetic pole, placed upon one of these lines, would travel along it away from the north pole of the field magnet and toward the south pole. An isolated south pole would move in an opposite direction. For many reasons it is desirable to direct the lines. Hence, as represented in Fig. 334, the direction which an isolated north pole would move is taken as the positive direction.

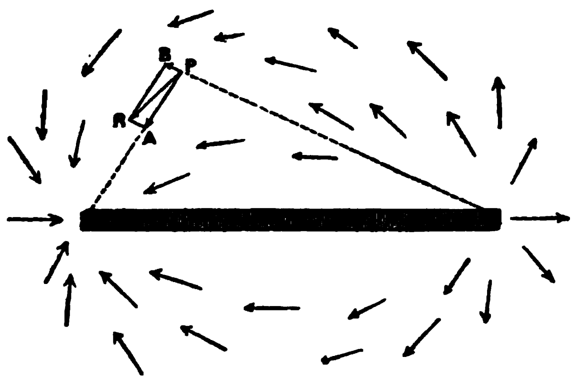
Of course it is impossible to get an isolated pole, but the undesired companion can be so far removed as not to interfere with demonstrative experiments. If a shallow glass dish, containing a little water, be placed over the magnet and spectrum shown in Fig. 335, and then a magnetized sewing-needle be floated in a vertical position, by means of a small cork, the lower pole of the needle will be so much nearer the magnet than the upper pole that it will act as an isolated pole. When placed over any line it will move along that line, however circuitous, until it reaches the pole of the field magnet, which attracts it. This experiment is much more satisfactory when the field magnet is an electro-magnet. The needle may then be placed at any desired point and commences to move only after the magnet is excited. Professor Spice sifts the filings upon a glass plate and projects the whole experiment from a vertical lantern.

If a short magnetic needle be moved around a field whose lines of force are graphically shown by iron-filings, the needle will turn until its magnetic axis coincides with the direction of the lines of force. In fact the filings themselves are little magnets, made so by induction, and tapping the paper upon which they rest serves the stead of a pivot. That the needle should so place itself is quite natural, for its north end tends to travel in one direction and its south end in an opposite direction. The result is a couple, which turns the needle until the pulls are from the same line of force, passing through the pivot.

605. Theory of the Curvature of the Lines.—In any plane passing through a magnet, *N*, *S* (Fig. 336), let *P* be an isolated unit north pole. Assume its distance from the north pole *PN*, to be twice as great as from the south pole *PS*. The unit will be repelled by the north pole with a certain force, which is represented in amount and direction by the line, *PB*. Then, according to the law of magnetic force (Art. 600), the attraction exerted by the south pole, which is only half as far away, will be four times as great, and is represented in magnitude and direction by the line *AP*. The resultant of these two forces must be the diagonal, *RP*, of the completed parallelogram, and the

unit pole would move along the line RP. If elementary parallelograms be constructed in this manner throughout the field,

FIG. 336.



their diagonals, when connected, will represent the lines of force.

The curvature, then, is the result of combined attraction and repulsion. The lines of magnetic force from an isolated pole would be straight, as are the lines along which gravitation acts, and the law given in Art. 600 is true for isolated poles only.

606. Fields from Several Magnets.—When several magnets are in the same vicinity, the resultant field, compounded from the separate fields of each magnet, is sometimes curiously arranged. Thus the field from two magnets, whose north and south poles are opposed to each other, is represented in Fig. 337. A short magnetic needle would be in stable equilibrium if placed in any part of this field. Fig. 338 shows quite a different field where the opposed poles are like named. A needle moved about this field would suddenly turn half round on its axis at the moment of crossing the line cd. When the pivot is exactly upon cd, the needle's south pole is attracted equally in opposite directions by the two exposed north poles. In the same manner its north pole is repelled.

FIG. 337.

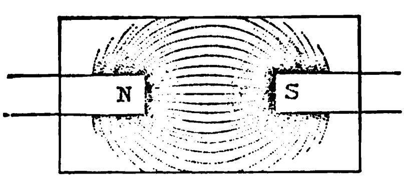
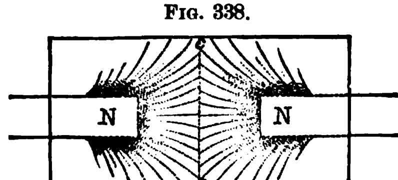


FIG. 338.



The result is that the needle places itself so that its axis is perpendicular to the axis of the field magnets.

Much may be learned by experimenting with iron-filings on variously compounded fields.

607. Strength or Intensity of Magnetic Field.—It may be reasonably supposed that each line of force exerts, along its length, a given amount of force. Hence a piece of iron, which was traversed by several lines, would be more powerfully attracted than if traversed by a fewer number. Thus a magnet pole of definite size, placed near to the pole of the field magnet, would be attracted with more force than at a distance, for the lines are closer together near the poles of the field magnet. The number of lines of force, then, which penetrate a given area, determines the relative force exerted by the field at that place. This is termed the *strength* or *intensity* of the field.

In order to compare the strengths of different fields, it is necessary to have a unit of strength. Hence

A magnetic field of unit strength is one which exerts a unit force (dyne) *upon a free unit magnet pole.*

608. Determination of the Strength of a Field.—If a magnet, suspended by a fibre, be placed in magnetic fields of different strengths, it will oscillate for a long time, and the times of oscillation will be shorter the stronger the field. This is parallel to the case of pendulum vibrations. The pendulum vibrates because of the force exerted by gravity and because of its inertia. Gravity pulls its centre of gravity as near as possible to the earth, and inertia carries it beyond this position. If the force of gravity were increased, the pendulum would vibrate quicker. In the case of a magnet, the force of the field takes the place of gravity. Now, just as the force of gravity can be measured by the time of oscillation of a given pendulum (Art. 163), so the strength of a magnetic field can be measured by the time of oscillation of a given magnet.

If $t =$ the time taken by the magnet in passing from one turning-point to the other in an oscillation, $K =$ the moment of inertia of the magnet (Art. 160), $M =$ the magnetic moment of the magnet, then the strength of the field

$$H = \frac{\pi^2}{t^2}\frac{K}{M}.$$

When the same magnet is used K and M are constant, hence

$$H \propto \frac{1}{t^2} \propto n^2,$$

where $n =$ the number of single vibrations in a second. Then if

a given magnet vibrates n and n' times per second in two fields of strengths H and H',

$$\frac{H}{H'} = \frac{n^2}{n'^2}.$$

If the values of M and K are known or determined, then the first equation gives the absolute strength of the field, provided all the quantities are expressed in proper units.

609. Hysteresis.—If a piece of iron be placed in a magnetic field, it will have two opposite poles induced in it whose strengths depend upon the strength of the field. If the strength of the field be varied from zero to a maximum, and then to zero again, there will be two times when the field will have a definite strength —once when the field is growing stronger and again when it is decreasing in strength. The strengths of the induced poles in the iron are different in these two equal fields. They will be less in the increasing field than in the decreasing. The strength of the iron's poles depends upon the iron's previous history. The iron has a tendency to remain in its previous condition and behind the field's requirements. This peculiarity of the iron is termed by Ewing *static hysteresis.*

If iron be placed in a magnetic field of constant strength, it will require a certain time before its induced poles assume constant strengths. To this property of the iron Ewing gives the name *viscous hysteresis.*

610. Number of Lines of Force from a Given Pole.—It is convenient to consider the number of lines of force passing through a given area in a field as the measure of the strength of the field. Each line may be supposed to exert a dyne of force on a unit pole pierced by it. The given area is a square centimetre and is placed so as to be perpendicular to the lines of force. *A unit field would* then *have one line passing through a square centimetre.*

Suppose now that, around a unit magnet pole, we conceive a spherical shell of one centimetre radius. From the definition of a unit pole (Art. 601) we know that the enclosed pole exerts, on another unit pole, a dyne of force at every point on this shell. The strength of the field, then, at all these points, is unity. Hence every centimetre of it is pierced by one line of force. But the whole surface of the sphere of unit radius contains 4 π centimetres. *The unit pole, therefore, sends off* 4 π *lines of force.* An enclosing surface of any size would be pierced by the same number of lines.

If the strength of the pole were 2 units, it would send off 8 π lines; or, in general,

A magnet pole, of strength m, sends out 4 π m lines of force.
This conception of the magnetic lines has recently developed into many important theoretical conclusions, which have equally important practical applications.

With a real magnet, having two poles, it is important to remember that the lines of *induction*,* starting out from one pole, finally arrive at the other pole and thence pass through the magnet itself. Hence the number passing through a section of the magnet lying midway between the poles is 4 π m.

611. Magnetic Susceptibility.—If two magnets, with their opposite poles opposed to each other, be arranged along a common axis, and if the lines of induction be made visible by iron-filings, the resulting spectrum will be as in Fig. 339. If, now, a piece of iron

FIG. 339.

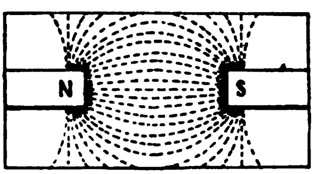
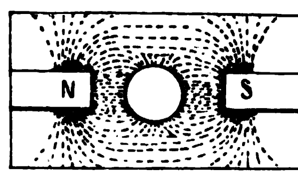


FIG. 340.



is placed between the poles, the field alters and will give, *e.g.*, the spectrum shown in Fig. 340. The lines are much more numerous where the iron is than they were before it was placed there. Had a piece of brass been used instead of iron, the field would have remained undistorted. A piece of cobalt would have produced some distortion, but not as much as the iron.

The cause of the additional lines is that the iron has, under the influence of the field, become an induced magnet and has added its lines to those already in the field.

If the field magnets were made stronger, they would send out more lines, and the iron would become more strongly magnetized and would also send out more lines. Thus, supposing that the iron does not become saturated, the strength of its pole depends upon the strength of the field and upon the volume of the iron. Suppose that the strength of the induced pole of a rectangular prism of iron is m; that the pole length equals the physical

* A line of induction differs from a line of force in that it does not change its direction on its return through the body of the magnet.

length of the iron, l; that the cross-section of the iron is s. Then the intensity of magnetization (Art. 601)

$$\mathbf{I} = \frac{m\,l}{s\,l} = \frac{m}{s}.$$

If the cross-section s be one centimetre, then $\mathbf{I} = m$, or the intensity of magnetization is equal to the strength of the induced pole. It has been found that if the strength of the field equals $\mathbf{H}$,

$$\mathbf{I} = k\,\mathbf{H},$$

where k is a constant called the *magnetic susceptibility*. It depends upon the kind of iron or other substance placed in the field. For iron, nickel, and cobalt the value is positive; for vacuum, air, and most gases is practically zero, and for bismuth, antimony, and phosphorus it is negative, though extremely small.

612. Magnetic Permeability.—In most all of the practical problems on magnetic induction it is desirous to know the total number of lines which pass through the iron suffering induction. In the iron prism of the preceding article the total number traversing it is made up of two parts: $4\,\pi\,\mathbf{I} = 4\,\pi\,k\,\mathbf{H}$ lines from the induced pole and $\mathbf{H}$ lines from the original field. Representing this sum by $\mathbf{B}$, we have

$$\begin{aligned}\mathbf{B} &= \mathbf{H} + 4\,\pi\,k\,\mathbf{H}\\ &= (1 + 4\,\pi\,k)\,\mathbf{H}.\end{aligned}$$

It is customary to place $1 + 4\,\pi\,k = \mu$, whence

$$\mathbf{B} = \mu\,\mathbf{H}.$$

Since μ involves k, it depends upon the character of the substance under induction. For air and gases it is unity; for iron, etc., greater than unity (sometimes reaching 16,000), and for bismuth, etc., less than unity.

As $\mathbf{B}$ represents the number of lines that pass through a square centimetre of iron, and $\mathbf{H}$ the number through air, then the iron may be said to conduct magnetic lines μ times better than air. From consideration in this light μ has received the name *magnetic permeability*.

The magnetic permeability of a substance is its relative conductivity for magnetic lines of force as compared with vacuum (or air) *as a standard.*

The equations connecting $\mathbf{B}$, $\mathbf{H}$, $\mathbf{I}$, μ, and k, which have been given are true whatever be the cross-section of the iron under induction. The assumption of a square centimetre cross-section is for simplification only.

613. The Magnetic Circuit.—In the construction of most electro-magnetic apparatus it is of utmost importance that *as much as possible of the field of the magnetizing agent shall be occupied by a*

substance of great permeability such as iron. For instead of having merely the lines which can be sent through air by the agent we can just as well have the additional ones from the iron. Of course an air gap must be left somewhere in the circuit of the lines in order to introduce the body to be acted upon. But this gap should be as small as possible if a maximum effect be desired.

If the opposite poles of two straight electro-magnets be caused to attract a piece of iron, the iron fills in one gap, but the lines from the other ends pass through the air. The force of the original attraction would be much increased if the extreme ends were connected by an iron bar. This last bar sends its additional lines through the magnets and increases the force.

614. Paramagnetism and Diamagnetism. — Substances which have a permeability greater than 1 (that of vacuum) as iron, steel, nickel, cobalt, etc., are attracted by a magnet and tend to move toward it. If not allowed to move toward, but allowed to rotate, they will tend to set themselves axially with the lines of induction. These are called *paramagnetic* substances.

Substances of permeability less than unity show the opposite tendencies. They are repelled by magnets and set themselves perpendicular to the lines of force. They are bismuth, antimony, phosphorus, etc. Without making use of the term permeability we may say:

Those substances which are attracted by a magnetic pole, or which in a magnetic field tend to move from places of less to places of greater intensity, are called Paramagnetic.

Those substances which are repelled by either pole indifferently, or which move from places of greater intensity to places of less intensity in the field, are called Diamagnetic.

In order to explain the phenomena of paramagnetism and diamagnetism we have to consider that the movable parts of a magnetic circuit strive to adjust themselves so that the maximum lines of induction shall pass through the circuit. Paramagnetic substances are thus drawn into the circuit and place themselves longitudinally with the lines, while diamagnetic substances act in an opposite manner, the air furnishing more lines than if they should displace it.

The repulsion of diamagnetic substances is hard to illustrate before a large audience. A huge electro-magnet may be made to slightly repel a piece of bismuth suspended on a long, delicate fibre. Better results can be obtained by approaching a large piece of bismuth to one of the needles in an astatic magnetometer (Art. 625).

CHAPTER IV.

TERRESTRIAL MAGNETISM.

615. The Earth a Magnet.—If a needle is carried round the earth from north to south, it takes approximately all the positions in relation to the earth's axis which it assumes in relation to a magnetic bar, when carried round it from end to end. At the equator it is nearly parallel to the axis, and it inclines at larger and larger angles as the distance from the equator increases; and in the region of the poles it is nearly in the direction of the axis. *The earth itself, therefore, may be considered a magnet*, since it affects a needle as a magnet does, and also induces the magnetic state on iron. But it is necessary, on account of the attraction of opposite poles, to consider the northern part of the earth as being like the south pole of a needle, and the southern part like the north pole.

616. Declination of the Needle.—When the needle is balanced horizontally, and free to revolve, it does not generally point exactly north and south; and the angle by which it deviates from the meridian is called the *declination*. A vertical circle coincident with the direction of the needle at any place is called the *magnetic meridian*. As the angle between the magnetic and the geographical meridians is generally different for different places, and also varies at different times in the same place, the word *variation* expresses these *changes* in declination, though it is much used as synonymous with declination itself.

The force which causes the needle to set in the magnetic meridian is *merely directive.*

If the needle be weighed before it is magnetized and again after it has been made a magnet, no change of weight can be detected, proving that the earth's attraction for one pole is exactly equal to its repulsion of the other. This may also be shown by attaching a magnet to a cork and thus floating it upon water. It will set in the magnetic meridian but will show no tendency to move across the water toward the north, nor in any other direction. This effect is due to the earth's uniform magnetic field. The magnetic pole of the earth being practically at an infinite distance, the forces of attraction and repulsion, being equal, constitute a couple.

617. Isogonic Curves.—This name is given to a system of lines imagined to be drawn through all the points of equal decli-

nation on the earth's surface. We naturally take as the standard line of the system that which connects the points of *no declination*, or the isogonic of 0° (Fig. 341). Commencing at the north

FIG. 341.

pole of dip, about Lat. 70°, Lon. 96°, it runs in a general direction E. of S., through Hudson's Bay, across Lake Erie, and the State of Pennsylvania, and enters the Atlantic Ocean on the coast of North Carolina. Thence it passes east of the West India Islands, and across the N. E. part of South America, pursuing its course to the south polar regions. It reappears in the eastern hemisphere, crosses Western Australia, and bears rapidly westward across the Indian Ocean, and then pursues a northerly course across the Caspian Sea to the Arctic Ocean. There is also a detached line of no declination, lying in eastern Asia and the Pacific Ocean, returning into itself, and inclosing an oval area of 40° N. and S. by 30° E. and W. Between the two main lines of no declination in the Atlantic hemisphere, the declination is *westward*, marked by continued lines in Fig. 341; in the Pacific hemisphere, outside of the oval line just described, it is *eastward*, marked by dotted lines. Hence, on the American continent, in all places east of the isogonic of 0°, the north pole of the needle declines westward, and in all places west of it, the north pole declines eastward; on the other continent this is reversed, as shown by the figure.

Among other irregularities in the isogonic system, there are two instances in which a curve makes a wide sweep, and then intersects its own path, while those within the loop thus formed return into themselves. One of these is the isogonic of 8° 40′ E., which intersects in the Pacific Ocean west of Central America; the other is that of 22° 13′ W., intersecting in Africa.

In the northeastern part of the United States the declination has long been a few degrees to the west, with very slow and somewhat irregular variations.

618. Secular and Annual Variation.—The declination of the needle at a given place is not constant, but is subject to a slow change, which carries it to a certain limit on one side of the meridian, when it becomes stationary for a time, and then returns, and proceeds to a certain limit on the other side of it, occupying two or three centuries in each vibration. At London, in 1580, the declination was 11¼° E.; in 1657, it was 0°; after which time the needle continued its western movement till 1818, when the declination was 24½° W.; since then the needle has been moving slowly eastward, and in 1879, at Kew, the declination was 19° 7′ *west.*

The entire secular vibration will probably last more than three centuries. The average variation from 1580 to 1818 was 9′ 10″ annually. But, like other vibrations, the motion is slowest toward the extremes.

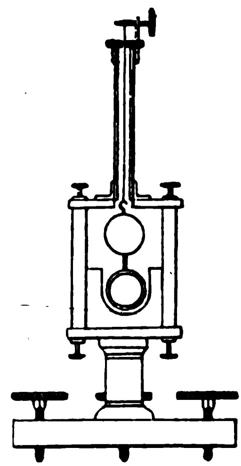
There has also been detected a small *annual* variation, in which the needle turns its north pole a few minutes to the east of its mean position between April and July, and to the west the rest of the year. This annual oscillation does not exceed 15 or 18 minutes.

619. Diurnal Variation.—The needle is also subject to a small *daily* oscillation. In the morning the north end of the needle has a variation to the east of its mean position greater than at any other part of the day. During winter this extreme point is attained at about 8 o'clock, but as early as 7 o'clock in the summer. After reaching this limit it gradually moves to the west, and attains its extreme position about 3 o'clock in winter, and 1 o'clock in summer. From this time the needle again returns eastward, reaching its first position about 10 P.M., and is almost stationary during the night. The whole amount of the diurnal variation rarely exceeds 12 minutes, and is commonly much less than that. These diurnal changes of declination are connected with changes of *temperature*, being much greater in summer than in winter. Thus, in England the mean diurnal variation from May to October is 10 or 12 minutes, and from November to April only 5 or 6 minutes.

620. Magnetometer.—In determining and observing the variation of the declination use is made of a magnetometer.

Fig. 342 represents such an instrument. It consists of a magnetized ring surmounted by a circular mirror, both being suspended by a silk fibre. The poles of the ring are at the sides and the plane of the ring, when at rest, coincides with the plane of the magnetic meridian. Any variation of the meridian is followed by a movement of the ring. The mirror, being connected with the ring, moves also. This small movement may be magnified and observed by means of a telescope and scale. The image of the scale is reflected from the mirror into the telescope.

FIG. 342.

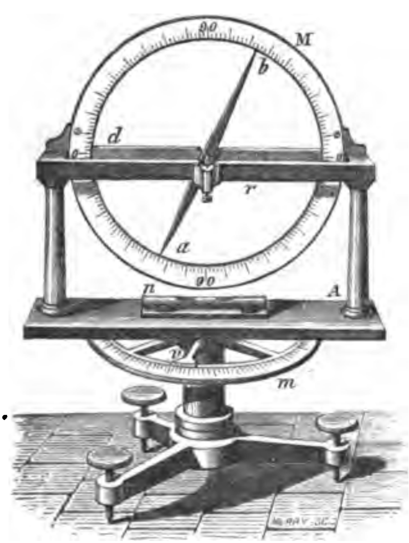


Surrounding the ring magnet is a hollowed-out piece of pure copper. This brings the magnet quickly to rest by means of the electrical currents induced in it (Art. 671) by the moving magnet.

Magnetometers are also used in determining the magnetic moment of bar magnets.

621. Dip of the Needle.—A needle first balanced on a horizontal axis, and then magnetized and *placed in the magnetic meridian,* assumes a fixed relation to the horizon, one pole or the other being usually depressed below it.

FIG. 343.



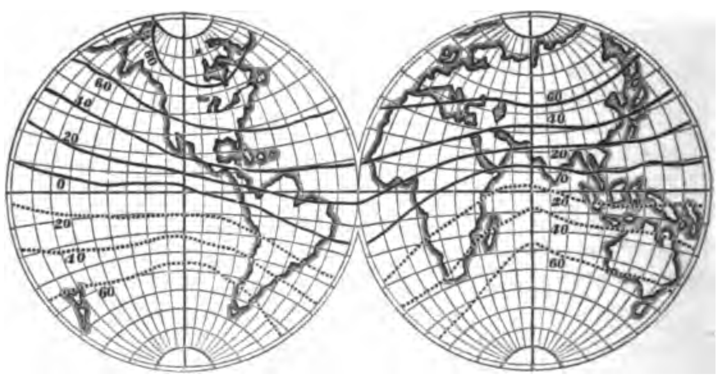
The axis of the needle must be placed very accurately at right angles to the plane of the magnetic meridian, or false indications will be given; if the axis of suspension were placed *in* the plane of the meridian the angle of depression would be 90° at all places on the earth's surface.

The angle of depression is called the *dip* of the needle. Fig. 343 represents the *dipping-needle,* with its adjusting screws and spirit-level; and the depression may be read on the graduated scale. After the horizontal circle *m* is

levelled by the foot-screws, the frame A is turned horizontally till the vertical circle M is in the magnetic meridian. For north latitudes, the north end of the needle is depressed, as a in the figure.

622. Isoclinic Curves.—A line passing through all points where the dip of the needle is nothing, *i.e.*, where the dipping needle is horizontal, is called the *magnetic equator of the earth.* It can be traced in Fig. 344 as an irregular curve around the

FIG. 344.



earth in the region of the equator, nowhere departing from it more than about 15°. At every place north of the magnetic equator the north-seeking pole of the needle descends, and south of it the south-seeking pole descends; and, in general, the greater the distance, the greater is the dip. Imagine now a system of lines, each passing through all the points of equal dip; these will be nearly parallel to the magnetic equator, which may be regarded as the standard among them. These magnetic parallels are called the *isoclinic curves;* they somewhat resemble parallels of latitude, but are inclined to them, conforming to the oblique position of the magnetic equator. In the figure, the broken lines show the dip of the south pole of the needle; the others, that of the north pole. The points of greatest dip, or dip of 90°, are called the *poles of dip.* There is one in the northern hemisphere, and one in the southern. The north pole of dip was found, by Captain James C Ross, in 1831, to be at or very near the point, 70° 14′ N.; 96° 40′ W., marked **x** in the figure. The south pole is not yet so well determined.

At the poles of dip the horizontal needle loses all its directive power, because the earth's magnetism tends to place it in a vertical line, and, therefore, no component of the force can operate in

a horizontal plane. The isogonic lines in general converge to the two dip-poles; but, for the reason just given, they cannot be traced quite to them.

The dip of the needle, like the declination, undergoes a variation, though by no means to so great an extent.

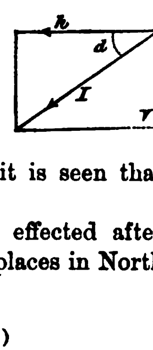
In 1576, the date of its discovery, the dip at London was 71° 50′; it increased to a maximum of 74° 42′ in 1723, since which time it has gradually decreased. In 1879 the dip at Kew was 67° 42′.

In the course of 250 years, it has diminished about five degrees in London. In 1820 it was about 70°, and diminishes from two to three minutes annually.

Since the dip at a given place is changing, it cannot be supposed that the poles are fixed points; they, and with them the entire system of isoclinic curves, must be slowly shifting their locality.

623. Intensity of the Earth's Magnetism.—The axis of the dip-needle, when placed in the magnetic meridian, coincides in direction with the lines of force of the earth's magnetic field. The magnetic force, then, acts in this inclined direction. In most magnetic determinations, however, the needle employed swings in a horizontal plane, and the force exerted upon it by the earth is only that portion of its total force which acts in a horizontal direction. This horizontal component of the strength of the field is called the *horizontal intensity* of the earth's magnetism. Let I (Fig. 345) represent the strength of the earth's field along the lines of force, *i.e.*, along the axis of the dip-needle, d = angle of dip, then h = the horizontal intensity. From the diagram it is seen that

FIG. 345.



$$h = I \cos d.$$

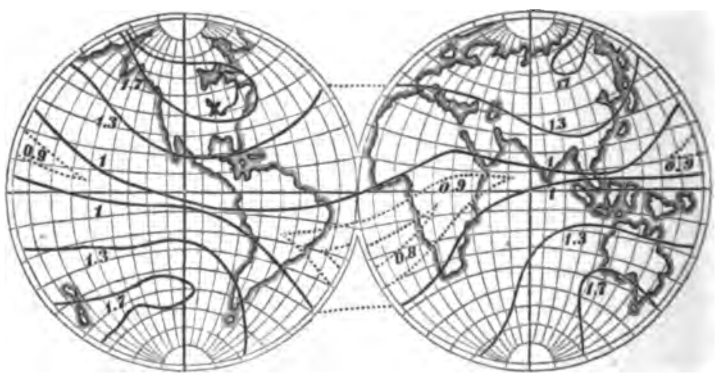
The determination of the horizontal intensity is effected after the manner described in Art. 608. Its values for places in North America are given in the following table:

HORIZONTAL INTENSITY. (C. G. S. UNITS.)

Boston	0.172
Cleveland	0.184
Chicago	0.184
Halifax	0.159
Montreal	0.147
New York	0.184
New Orleans	0.281
Niagara	0.167
Philadelphia	0.194
San Francisco	0.255
Washington	0.200

624. Isodynamic Curves.—An inspection of the table just given shows that the horizontal intensity increases as we near the equator. The strength of the earth's field in the direction of its lines of force, however, decreases on nearing the equator, as might be expected, the equator being farthest from the poles. After ascertaining, by actual observation, the intensity of the magnetic force in different parts of the earth, lines are supposed to be drawn through all those points in which the force is the same; these lines are called *isodynamic curves*, represented in Fig. 346. These

FIG. 346.



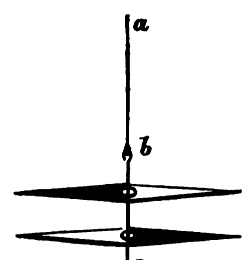
also slightly resemble parallels of latitude, but are more irregular than the isoclinic lines. There is no one standard equator of minimum intensity, but there are two very irregular lines surrounding the earth in the equatorial region, in some places almost meeting each other, and in others spreading apart more than two thousand miles, on which the magnetic intensity is the same. These two are taken as the standard of comparison, because they are the lowest which extend entirely round the globe. The intensity on them is therefore called *unity*, marked 1 in the figure. In the wide parts of the belt which they include—lying one in the southern Atlantic, and the other in the northern Pacific oceans—there are lines of lower intensity which return into themselves, without encompassing the earth. In approaching the polar regions, both north and south, the curves, retaining somewhat the form of the unit lines, are indented like an hour-glass, as those marked 1.7 in the figure, and at length the indentations meet, forming an irregular figure 8; and at still higher latitudes, are separated into two systems, closing up around two poles of maximum intensity. Thus there are on the earth four poles of maximum intensity, two in the northern hemisphere and two in the southern. The Ameri-

can north pole of intensity is situated on the north shore of Lake Superior. The one on the eastern continent is in northern Siberia. The ratio of the least to the greatest intensity on the earth is about as 0.7 to 1.9; that is, as 1 to 2$\frac{5}{7}$. In the figure, intensities less than 1 are marked by dotted lines.

625. Variation in the Strength of the Earth's Field.—Astatic Needles.—The intensity of the earth's magnetism is constantly changing. These changes consist in small fluctuations about an average constant strength. Many electrical determinations require for their accuracy either that the horizontal intensity should remain constant or that its fluctuations should be taken into account. As the latter is the only alternative, a means must be had of determining, at any moment, whether the intensity has changed, and, if so, how much.

One method is to employ a magnetometer (Fig. 342), which is rendered nearly *astatic* by a supplementary bar magnet. (A needle is astatic when the earth has no directive effect upon it.) This auxiliary magnet is placed north and south, directly under, or over, the needle of the magnetometer. When placed at a proper distance above the needle, depending upon its strength, it will act upon the needle with the same force as the earth, only in an opposite direction. It will thus neutralize the influence of the earth and the needle can turn into any position. If the magnet be brought a little nearer, the needle will suddenly turn around and its north-seeking pole will point south. Now, by a little delicate manipulation, the needle may be made to point nearly east and west. In this position it is very sensitive. Any small increase in the earth's intensity will cause its north-seeking end to turn to the north, and any decrease to the south. Thus, by looking through the telescope at the mirror, any change in the intensity can be detected at any moment, and the amount of change can be arrived at by calculation.

FIG. 347.



Astatic needles are of great value in electrical measurements. Liberated from the earth's directive action they may still be affected by electrical currents. Another method of obtaining this end is shown in Fig. 347.

A compound needle, consisting of two simple needles fixed upon a wire, with their unlike poles opposed, may be suspended in any of the usual modes. If the needles are exactly equal in *all respects* the system

will be perfectly astatic. The condition of perfect equality in all the conditions is never realized.

626. Magnetic Charts.—These are maps of a country, or of the world, on which are laid down the systems of curves which have been described. But for the use of the navigator, only the isogonic lines, or lines of equal declination, are essential. There are large portions of the globe which have as yet been too imperfectly examined for the several systems of curves to be accurately mapped. It must be remembered, too, that the earth is slowly but constantly undergoing magnetic changes, by which, at any given place, the declination, dip, and intensity are all essentially altered after the lapse of years. A chart, therefore, which would be accurate for the middle of the nineteenth century, will be, to some extent, incorrect at its close.

627. The Declination Compass.—This instrument consists of a magnetic needle suspended in the centre of a cylindrical brass box covered with glass; on the bottom of the box within is fastened a circular card, divided into degrees and minutes, from 0° to 90° on the several quadrants. On the top of the box are two uprights, either for holding sight-lines or for supporting a small telescope, by which directions are fixed. The quadrants on the card in the box are graduated from that diameter which is vertically beneath the line of sight.

When the axis of vision is directed along a given line, the needle shows how many degrees that line is inclined to the magnetic meridian. In order that the angle between the line and the geographical meridian may be found, the declination of the needle for the place must be known.

628. The Mariner's Compass.—In the mariner's compass (Fig. 348) the card is made as light as possible, and attached to the needle, so that the north and south points marked on the card always coincide with the magnetic meridian. The index, by which the direction of the ship is read, consists of a pair of vertical lines, diametrically opposite to each other, on the interior of the box. These lines, one of which is seen at *a*, are in the plane of the ship's keel. Hence, the degree of the card which is against either of the lines shows at once both the angle with the magnetic meridian and the quadrant in which that angle lies.

FIG. 348.

In order that the top of the box may always be in a horizontal position, and the needle as free as possible from agitation by the rolling of the ship, the box, B, is suspended in *gimbals*. The pivots, A, A, on opposite sides of the box, are centred in the brass ring, C, D, while this ring rests on an axis, which has its bearings in the supports, E, E. These two axes are at right angles to each other, and intersect at the point where the needle rests on its pivot. Therefore, whatever position the supports, E, E, may have, the box, having its principal weight in the lower part, maintains its upright position, and the centre of the needle is not moved by the revolutions on the two axes.

On account of the dip, which increases with the distance from the equator, and is reversed by going from one hemisphere to the other, the needle needs to be loaded by a small adjustable weight, if it is to be used in extensive voyages to the north or south.

629. Aurora Borealis.—This phenomenon is usually accompanied by a disturbance of the needle, thus affording visible indications of a magnetic storm; but the contrary is by no means generally true, that a magnetic storm is accompanied by auroral light. The connection of the aurora borealis with magnetism is manifested not only by the disturbance of the needle, but also by the fact that the streamers are parallel to the dipping-needle, as is proved by their apparent convergence to that point of the sky to which the dipping-needle is directed. This convergence is the effect of perspective, the lines being in fact straight and parallel.

630. Why is the Earth a Magnet?—Modern discoveries in electro-magnetism and thermo-electricity furnish a clew to the hypothesis which generally prevails at this day. Attention has been drawn to the remarkable agreement between the *isothermal* and the *isomagnetic* lines of the globe. The former descend in crossing the Atlantic Ocean toward America, and there are two poles of maximum cold in the northern hemisphere. The isoclinic and the isodynamic curves also descend to lower latitudes in crossing the Atlantic westward; so that, at a given latitude, the degree of *cold*, the magnetic *dip*, and the magnetic *intensity*, are each considerably greater on the American than on the European coast. This is only an instance of the general correspondence between these different systems of curves. It has likewise been noticed (Art. 619) that the needle has a movement diurnally, varying westward during the middle of the day, and eastward at evening, and that this oscillation is generally much greater in the hot season than the cold. It is obvious, therefore, that the development of magnetism in the earth is intimately connected with the tempera-

ture of its surface. Hence it has been supposed that the heat received from the sun excites electric currents in the materials of the earth's surface, and these give rise to the magnetic phenomena.

Most interesting is the hypothesis recently projected by Professor Bigelow, viz., the earth is revolving and moving in a magnetic field, which is created by the sun. According to this the earth is a magnet by induction and the variations in its magnetism are caused by differences in the strength of the field through which it is moving.

CHAPTER V.

CURRENT ELECTRICITY.

631. Electricity in Motion.—It has been seen (Art. 571) that when conductors which have a difference of electrical potential are connected together by a conducting substance, a flow, or *current*, of electricity from the higher to lower potential takes place. This current, however, lasts for an instant only, and any phenomena due directly to the flow would have to be observed during that instant. If by any means the difference of potential of the bodies could be maintained in spite of their being connected, a continuous current would be made to traverse the connecting conductor. Such a means was accidentally discovered in 1786, by Galvani, Professor of Anatomy at Bologna. After experimenting, one day, upon the effects of statical electricity on a frog's leg, he hung the moist leg, by means of a copper hook, upon an iron window-guard. He then noticed that, whenever the free end of the leg touched the guard, it gave a spasmodic twitch, as though a statical charge had been passed through it. He accordingly surmised that he had found a new method of obtaining electricity.

632. Galvanic Cells.—Galvani's discovery has developed into the *Galvanic Cell* or *Element*—an arrangement of apparatus designed to give a continuous flow of electricity.

If, when two different substances are submerged in an oxidizing fluid,* one of them has a greater affinity for oxygen than the other, then a difference of potential will be set up between them. The one having the greater affinity will have a lower potential. If desirable, the substances may be considered as having become electrified—the least oxidizable positively, and the other negatively.

* For simplicity, affinity for oxygen is alone mentioned here. The principle is true for any chemical affinity.

If, now, the substances be connected by a wire, a current will flow through it from the higher to the lower potential. As long as the chemical action keeps up, the difference of potential and the current resulting from it will be maintained.

If, for example, the two substances were copper and zinc plates, and the fluid was dilute sulphuric acid (Fig. 349), the zinc, having greater affinity for the oxygen of the acid, would have a lower potential than the copper. Upon connecting them by a wire, a current would flow from the copper to the zinc. This would be a simple galvanic element.

FIG. 349.

The arrangement need not be as shown in the figure, for a zinc rod, wrapped in blotting-paper, upon which is wound bare copper wire, would, upon moistening the paper with dilute acid, give a current.

The flow from copper to zinc, in the connecting wire, is always accompanied by an equal flow from zinc to copper through the submerging fluid. (This latter flow is found necessary for the maintenance of the potential difference.) Thus, if we start at any point and follow the current, we will eventually come back to the point whence we started, *i.e.*, *a current of electricity always flows in a closed circuit.*

633. Electromotive Force.—The difference of potential set up in a galvanic element is due to an *Electromotive Force*, which is generally represented by the letters E. M. F. Its amount depends upon the nature of the two substances employed—their relative affinities for the active part of the fluid. In dilute sulphuric acid they arrange themselves in the following order:

Hydrogen,
Zinc,
Iron,
Lead,
Nickel,
Bismuth,
Copper,
Carbon,
Silver,
Platinum,
Oxygen.

Of the metals given, zinc has the greatest affinity for oxygen, and platinum the least. These two metals then would give the

greatest E. M. F. Platinum and silver would give hardly any. If two elements be constructed, using lead-zinc for one and lead-copper for the other, the current would flow out of the lead in the first case, and into the lead in the second.

The absolute electrostatic unit of potential difference is too large for practical purposes, hence a practical unit of E. M. F., called the *volt* (= $\frac{1}{300}$ electrostatic unit) is employed. The E. M. F. of copper-zinc in dilute sulphuric acid, at the instant of making first contact, is 0.921 volt.

The E. M. F. of a cell is independent of the size of the electrodes.

A copper-zinc cell of 1 sq. cm. electrodes has the same E. M. F. as one with 1,000 sq. cms.

The total E. M. F. in a circuit is equal to the algebraic sum of the separate E. M. F.'s.

Thus, if two copper-zinc cells be connected in a circuit in the order (Cu—Zn)—(Zn—Cu) one will tend to send a current in one direction, and the other in the opposite direction. The result will be no current at all.

If, in trying this experiment, one of the cells be very large and the other very small, the fact that no current flows would illustrate the fact that the E. M. F. is independent of the size of electrodes.

634. Polarization.—If a copper-zinc sulphuric-acid cell be connected with an electric bell (or any other current indicator) it will at first ring loudly, but will soon weaken, and finally cease to give a sound. Upon investigating the cause of this weakening it will be found that the E. M. F. has fallen from 1 volt to possibly 0.2 volt. This is because the current, which the element has sent through its own liquid, has decomposed that liquid, and hydrogen (Art. 678) has been deposited upon the copper and oxygen upon the zinc. The oxygen immediately enters into chemical union with the zinc, but the hydrogen remains in its gaseous form. The hydrogen, from its affinity for oxygen, sets up a *counter E. M. F.*, tending to send a current in an opposite direction. The resulting current is smaller than at first, and the cell is said to have become *polarized.*

635. Types of Batteries.—(A collection of galvanic cells is termed a *battery.*) Practical cells, designed for giving a constant flow of electricity, employ different methods for avoiding the counter E. M. F. of polarization. The market affords a great variety, but we need consider but three:

BUNSEN'S CELL.—As has been shown, the counter E. M. F. is due to hydrogen upon the electrode having the higher potential.

In Bunsen's cell this hydrogen is made to combine with oxygen furnished by nitric acid. The cell (Fig. 350) employs two different acids, which are kept separate by a porous, unglazed cup. This allows the electricity to flow, but prevents a free mixture of the acids. Outside the cup is zinc in dilute sulphuric acid; inside is carbon in nitric acid. The hydrogen, which above was deposited upon the copper, now comes off at the carbon. Instead of being allowed to exert a counter E. M. F., it is immediately oxidized by the nitric acid. On the other hand, this acid is prevented by the porous cup from violently attacking the zinc.

FIG. 350.

This cell has an E. M. F. of 1.8 volt, and is capable of maintaining it for a long time. It is a disagreeable cell to work with because of the nitric-acid fumes. These fumes can be avoided by substituting a solution of bichromate of potash for the nitric acid. It is also an active oxidizer; but, in time, large crystals form inside the walls of the porous cup and cause them to break in pieces.

DANIELL'S CELL.—This cell is more used than any other, in the laboratory. It also employs two liquids and a porous cup. The arrangement is (Fig. 351) zinc in dilute sulphuric acid inside the cup, and copper in copper sulphate outside. The cell's own current, instead of depositing hydrogen upon the copper, deposits copper from its sulphate. Now copper upon copper cannot alter the E. M. F. of the cell, and hence the Daniell has the most constant E. M. F. of ordinary cells. The E. M. F. depends somewhat upon the dilution of the sulphuric acid, but is very nearly 1 volt for any arrangement.

FIG. 351.

A modified form of Daniell's cell, called the *gravity cell*, is used in telegraphy. The porous cup is dispensed with, and the two liquids are kept separate by the action of gravity. The dilute sulphuric acid is floated on top of the copper sulphate.

LECLANCHE CELL.—There are more of this form of cell in use than of all others put together. They are not designed to main-

tain a constant E. M. F. for any great length of time. They are intended for purposes where a current is needed for only a few moments at most, as for electric bells or on telephone circuits. After use they regain their original E. M. F. The arrangement (Fig. 352) is zinc and carbon in a solution of sal ammoniac (NH_4Cl). An attempt is made to get rid of the hydrogen of polarization by surrounding or mixing the carbon with an oxide of manganese. This eventually oxidizes the hydrogen, but not as rapidly as the nitric acid in Bunsen's cell. Some forms of Leclanche cell employ a porous cup containing the carbon amid the manganese oxide. The E. M. F. of a fresh Leclanche is 1.5 volt.

FIG. 352.

636. Combustion of Zinc.—Nearly all batteries employ zincs for the lower potential electrode. A current flow is always accompanied by an oxidation of zinc, and the energy which accompanies the current comes from this oxidation. This is parallel to the case of a steam-engine, where the energy comes from the oxidation of the fuel under the boiler.

637. Amalgamation of Zincs.—Ordinary commercial zinc is impure. If this impurity were, say copper, and a particle should be embedded near the surface of a zinc electrode, then, upon immersing in acid, the zinc and copper would form a small cell by themselves. This would be giving a current, whether the complete cell were in use or not, and would be continually wasting zinc. This wasting of zinc, because of impurities in it, is called *local action* of the cell.

It has been found that local action can be prevented by amalgamating the zinc. This is done by dipping the zinc in acid and then in mercury. The mercury unites with the zinc and *floats* the impurities to its surface. These are then detached by the gas bubbles, which are caused by their union with the acid. The zinc of the amalgam is oxidized by the action of the battery, but the mercury remains unaffected. It is, therefore, constantly going

into combination with new zinc, as the action of the battery continues.

638. Practical Units of Current and Quantity.—As, in considering the flow of water in a pipe, we give the current a definite value of say so many gallons per hour, so we can give a definite value to the electrical current.

The quantity of water passing through any cross-section of a water-pipe of varying diameter is the same for the same time and current. Likewise

The quantity of electricity passing in the unit time through any cross-section of a simple undivided circuit is the same for the same current.

To obtain a unit for current we have only to use the unit for quantity and the one for time. Now, the absolute electrostatic unit of quantity is not of convenient size for practical purposes. Hence, a new unit, termed the *Coulomb*, is employed. It equals 3,000,000,000 absolute electrostatic units. We have, then,

The practical unit of current, the ampere, is that current which delivers one coulomb per second to any cross-section of the circuit.

639. Resistance.—All substance offers a resistance to the flow of electricity. Just as motion against resisting friction produces heat, so a current overcoming electrical resistance produces heat.

The resistance offered by a given conductor depends upon two things, viz., the character of the substance and its shape. If we represent the length of a conductor by l metres, its cross-section by q sq. mm., then its electrical resistance

$$R = s\,\frac{l}{q},$$

s being a constant depending upon the character of the substance and termed its *specific resistance.** If we assumed s, l, and q each equal to unity, we would have a unit resistance. A unit, much used in Germany, the Siemen's quicksilver unit, is defined by assuming that s for quicksilver at 0° C. is unity. Hence, *Siemen's unit of resistance is the resistance offered by a column of quicksilver* 1 *metre long and* 1 *sq. mm. cross-section, at* 0° C.

The international practical unit, the *legal ohm*, is a little larger than the Siemen's unit.

$$1\ ohm = 1.06\ Siemen's\ unit.$$

If R represents the resistance of a conductor, $^1/_R$ evidently represents its *conductivity*—the greater the resistance the smaller the

* The *absolute specific resistances* depending upon l and q being measured in centimetres, and R in absolute units, are expressed by unwieldy numbers, and a comprehension of the subject does not require their consideration.

conducting power. Accordingly, $^1/_s$ can be called the specific conductivity of a substance.

SPECIFIC CONDUCTIVITIES, $k = {}^1/_s$.

Mercury	1.06
Silver	63.
Copper	58.
Iron	7.4 to 9.5
Platinum	6.9
German silver	2.5 to 6.4
$Zn\,SO_4$ (sat. sol.)	.0000043
Pure Water	.000000000025
Glass	.0

These figures evidently represent the length in metres of a wire of 1 sq. mm. cross-section, that the resistance may be 1 ohm.

Their application can be best understood by an example. Determine the resistance of a copper wire 11.6 m. long and 0.1 sq. mm. in cross-section.

$$R = \frac{l}{k\,q} = \frac{11.6}{58 \times 0.1} = 2 \text{ ohms.}$$

Silver and copper are the best conductors we have. Because of the expense of the former, copper is universally employed on electrical circuits. In fact, some modern copper is said to conduct better than silver. Absolutely pure water is probably a non-conductor. The purest water yet obtained, if placed in a tube of unit diameter and 1 mm. long would offer the same resistance as a copper wire of same diameter, but as long as the orbit of the moon.

The influence of specific conductivity upon resistance can be prettily shown by the following experiment: Pass the current from a dynamo through an electric lamp, and, by means of two electrodes, through a vessel of rain-water. As long as the water is pure the lamp will not be illuminated. Place a few drops of sulphuric acid in the water, and the lamp will instantly commence to glow.

640. Influence of Temperature.—The resistance of conductors changes with the temperature. In all metals an increase of temperature increases the resistance. At ordinary temperatures the increase for most pure metals is 0.004 of the whole, per degree centigrade. The amount for German silver is about 0.0003.

Carbon and liquids decrease in resistance when the temperature is raised. The change per degree for liquids is between two and three per cent.

The dependence of resistance upon temperature furnishes a means of measuring the latter. A conductor of large temperature

coefficient is subjected to the heat whose temperature is to be determined, and while still in place its resistance is measured. The increase of resistance furnishes data for calculation of the temperature. By this means Professor Langley has measured the heat radiated from the moon.

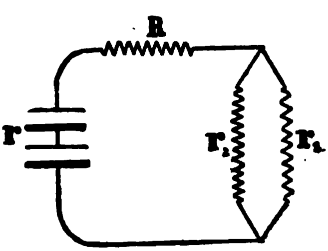
641. Ohm's Law.—The three electrical magnitudes—current, E. M. F., and resistance—are connected together by an important relation called Ohm's law. Letting E represent, in volts, the algebraic sum of all the E. M. F.'s of a circuit, R the sum of all the resistances, in ohms (of battery, conducting wires, and all instruments in circuit), then this law states, the current strength in amperes,

$$C = \frac{E}{R}.$$

The strength of the current varies directly as the E. M. F., and inversely as the resistance.

642. Divided Circuits.—Shunts.—If a current from a battery of E. M. F. $= E$ and *internal resistance* $= r$ be sent through a wire which divides at a certain point (Fig. 353) into two branches, which however reunite further on, and if the resistances of the undivided conductor and its branches are R, r_1, r_2 respectively, then the substitution of $r + R + r_1 + r_2$ in Ohm's law would not give the correct current strength. The reason for this is that the whole current does not pass through each of the branches r_1 and r_2. They each take a portion of the current, depending upon their resistances. A single conductor might be found, which, if substituted for the two, would leave the current in R unchanged. The resistance of this single conductor might be called the *equivalent resistance* of the branches. To determine this equivalent resistance it is most convenient to consider the *conductivities* of the branches Evidently the conductivity of the single replacing conductor must equal the sum of the conductivities of the separate paths. But the conductivities are the reciprocals of the resistances. Hence we have

FIG. 353.



$$\frac{1}{R'} = \frac{1}{r_1} + \frac{1}{r_2} \therefore R' = \frac{r_1 r_2}{r_1 + r_2}.$$

The current C flowing through the undivided portion of the circuit, *e.g.*, through R, would be, by Ohm's law,

$$\frac{E}{R + r + R'} = \frac{E}{R + r + \frac{r_1 r_2}{r_1 + r_2}}.$$

In the same manner the equivalent resistance of any number of different paths may be determined.

When a conductor is placed so as to take a portion of the current which is passing through another conductor it is called a *shunt* and the current is said to be shunted.

Many delicate instruments for measuring electrical quantities would be ruined if the whole current passed through them. In such cases a portion of the current is shunted off from the instrument. From the known resistances of the instrument and the shunt the quantities to be determined can be calculated.

643. Ratio of Currents in Shunts.—In order to determine the portion of the current flowing in any branch of a divided circuit, we must consider that the whole current is carried by the branches as a whole. Letting C = current in undivided portion (Fig. 353) and c_1 and c_2 = currents in r_1 and r_2, we have

$$C = c_1 + c_2.$$

Again, bearing in mind that the difference of potential (E. M. F.) between the ends of each branch is the same $= e$, we have, by Ohm's law,

$$c_1 = \frac{e}{r_1};\ c_2 = \frac{e}{r_2}, \text{ etc.}$$

$$\therefore\ c_1 : c_2 : \text{etc.} = \frac{1}{r_1} : \frac{1}{r_2} : \text{etc.}$$

The currents carried by different branches between two points of a circuit are inversely as the resistances of the branches.

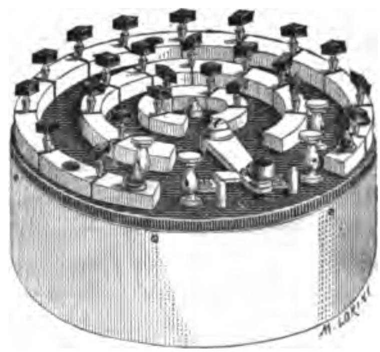
644. Fall of Potential.—If we have a battery connected through a uniform straight wire of given resistance, we may consider the potential at the zinc end to be = zero, and that at the other end positive, and = E. M. F. of the battery. Now, inasmuch as the resistances of equal lengths of the wire are the same, the potential at the middle of the wire equals one-half the E. M. F. At one-quarter the distance from each end of the wire the potentials are one-quarter and three-quarters of the E. M. F. The potential varies all along the wire, from zero at the zinc end to E. M. F. at the other end. If we commence at the other end we may say that

The potential falls directly as the resistance.

Of course, if the conductor were not homogeneous, *e.g.*, made

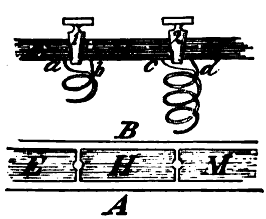
of copper and then German silver, the fall would not be the same for the same lengths. It would be more rapid in the German silver portion of the circuit than in the copper portion.

FIG. 354.



645. Resistance Boxes or Rheostats. — These are boxes (Fig. 354) containing different spools of wire, whose resistances have been determined, and which can be used for standards of comparison. German silver wire is generally used, because its resistance changes least with the temperature. The proper length of wire is taken, and, after being doubled at its middle (Fig. 355), is wound upon a spool, the two parts being wound side by side. This is indicated in the figure. The reason for this doubling is to avoid the self-induction disturbances of the wire (Art. 668). The spool, when wound, is placed inside the box and the terminals are fastened to two separate brass blocks on the top of the box. To each of these blocks is fastened one end of the two neighboring coils. In the figure, *a* and *b* of one resistance are fastened to blocks *E* and *H*. The ends *c* and *d* of the neighboring coil are fastened, one to *H* and the other to *M*. The blocks can be connected together at will by brass plugs fitting into holes between them.

FIG. 355.



Suppose, now, that the two terminals of a battery be connected with *E* and *M* respectively. If the plugs 1 and 2 be removed, the current will be obliged to traverse both of the resistance coils. If plug 1 be inserted, the current will divide between the plug and the coil. But the resistance of the plug is infinitesimal, and hence, practically, the whole current passes through it and none through the coil.

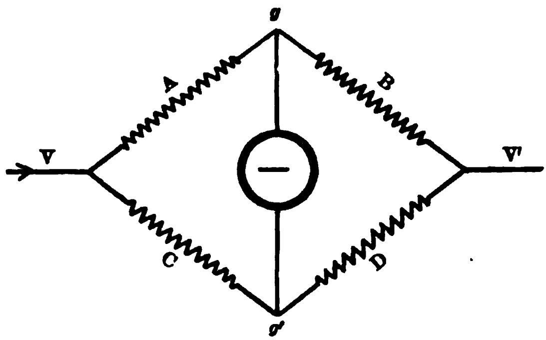
When a box of coils is inserted in a circuit, the resistance can be varied at will, by simply inserting or pulling out plugs.

646. Wheatstone's Bridge.—An important application of the preceding principles is Wheatstone's bridge. It is an arrange-

ment of apparatus by means of which resistances can be very accurately determined.

If a current of electricity arriving at V (Fig. 356) divides and

FIG. 356.



passes by two paths, $A\ B$ and $C\ D$, to V', where the paths unite, then the difference of potential ($V - V'$) is acting on both paths. The potential along each path must fall from V to V'. If at any point of the path $A\ B$, the potential is g, then some point (g') of the other path, $C\ D$, can be found having the same potential. If these two points, g and g' be connected through a galvanometer or any other current detector, no current will flow, because there is no potential difference between g and g'.

Inasmuch as the fall or loss of potential along each path is proportional to the resistance, the loss in passing A must be the same as in passing C, and the resistance of A must bear the same ratio to $A + B$ as C does to $C + D$. In order that the potentials may be the same at g and g', it must be true that the resistances follow the proportion

$$A : B = C : D.$$

In Wheatstone's bridge, when no current passes through the galvanometer, *the products of the opposite resistances are equal.*

The method of determining resistances by the bridge is to place the unknown resistance in one *arm* of the bridge, as D. Known resistances are placed in A and B, and a resistance-box in C. By manipulating the plugs in C a balance can be made so that no current flows through the galvanometer. A, B, and C are known, and the required resistance

$$D = \frac{B \times C}{A}.$$

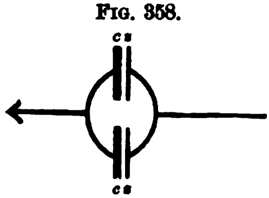
It is sometimes convenient to make C constant, and vary both A and B until a balance is obtained, A and B consisting of parts of the same straight wire of uniform diameter. The balance is

obtained by sliding the contact (g) with the galvanometer along this wire. The resistances of A and B are then proportional to their lengths.

647. Cells in Series and in Multiple Arc.—A battery of two cells can be connected to a circuit in two different manners. The copper of one may be connected to the zinc of the other (Fig. 357), and the circuit connected to the remaining zinc and copper. The cells would then be *in series*. Again, the coppers of each and the zincs of each might be connected together and the circuit connected to these short connecting wires (Fig. 358). The two cells are then said to be in *multiple arc*. Let us consider the results of these different arrangements. Represent the E. M. F. of each cell by E, and the internal resistance by r.

FIG. 357.

FIG. 358.



Evidently, when in series, the E. M. F. of the circuit is equal to the sum of the two E's, and the internal resistance of the battery is $2r$. If the resistance of the total external circuit be R, then the current, when the cells are in series,

$$C = \frac{2E}{R + 2r}.$$

When the cells are in multiple arc, the E. M. F. is no greater than for a single cell. The two cells are like a single cell of twice the size, and size does not affect the E. M. F. (Art. 633). The resistance, however, is only half that of a single cell, because the cross-section of the liquid, which the current has to pass, is twice as great. For two cells in multiple arc, then, the current

$$C = \frac{E}{R + \frac{r}{2}}.$$

We can extend this reasoning to any number of cells, and say, *n cells, in series, multiply the E. M. F. and the internal resistance by* n, and *m cells, in multiple arc, divide the internal resistance by m, but leave the E. M. F. unaltered, it being that of a single cell.*

In general, if we have $m\,n$ cells, consisting of n groups in series, each group containing m cells in multiple arc, the resulting current will be

$$C = \frac{nE}{R + n\frac{r}{m}}.$$

With a given number of cells and a given external resistance some arrangement of the cells can be found which will give a maximum current. It can be proved that this arrangement will render *the internal resistance as nearly equal to the external as possible.*

When the external resistance is very great compared with the battery, it is advisable to get as much E. M. F. as possible. This is accomplished by placing the cells in series.

Problems.

1. An incandescent lamp takes a current of 0.7 ampere, and the E. M. F. between its terminals is found to be 98 volts: what is its resistance?

2. A current of 8.5 amperes flows through a conductor, the ends of which are found to have a difference of potential of 24 volts: what is its resistance? *Ans.* 2.823 ohms.

3. A battery, arranged in series, consists of 5 Daniell cells, each having an E. M. F. of 1.08 volt and an internal resistance of 4 ohms: what current will the battery produce with an external resistance of 7 ohms. *Ans.* 0.2 ampere.

4. Two cells of E. M. F., 1.8 volt and 1.08 volt respectively, are placed in circuit in opposition (*i.e.*, with their poles in such positions that the cells tend to send currents in opposite directions). The current is found to be 0.4 ampere: what current will be produced, if the cells are placed properly in series?

Ans. 1.6 ampere.

5. A Bunsen cell has an internal resistance of 0.3 ohm and its E. M. F. on open circuit is 1.8 volt. The circuit is completed by an external resistance of 1.2 ohm: find the current produced and the difference of potential which now exists between the terminals of the cell.

$$Ans. \begin{cases} C = 1.2 \text{ ampere.} \\ P.\,D. = 1.44 \text{ volt.} \end{cases}$$

6. Two wires of the same length and material are found to have resistances of 4 and 9 ohms respectively: if the diameter of the first is 1 mm., what is the diameter of the second?

7. The resistance of a bobbin of wire is measured and found to be 68 ohms: a portion of the wire 2 metres in length is now cut off, and its resistance is found to be 0.75 ohm. What was the total length of wire on the bobbin? *Ans.* 181.3 metres.

8. What length of platinum wire 1 mm. in diameter is required in order to make a 10 ohm resistance coil?

9. A wire m metres in length and $1/n$th of a millimetre in diameter is found to have a resistance r: what is the specific resistance of the material of which it is made?

10. A uniform wire is bent into the form of a square: find the resistance between two opposite corners in terms of the resistance of one of the sides.

11. Twelve incandescent lamps are arranged in parallel between two electric light leads. The difference of potential between the leads is 99 volts, and each lamp takes a current of 0.75 ampere: what is the equivalent resistance between the leads?

Ans. 11 ohms.

12. A battery of 20 ohms resistance is joined up in circuit with a galvanometer of 10 ohms resistance. The galvanometer is then shunted by a wire of the same resistance as its own: compare the currents produced by the battery in the two cases.

Ans. $C : C' = 5 : 6$.

13. In the preceding example determine the ratio between the currents which flow through the galvanometer before and after it is shunted.

14. How would you arrange a battery of 12 cells, each of 0.6 ohm internal resistance, so as to send the strongest current through an electro-magnet of resistance of 0.7 ohm.

15. In a Wheatstone's bridge (Fig. 356) $A = 10$ ohms, $B = 1{,}000$ ohms, and $C = 50$ ohms: what is the resistance of D, if the galvanometer shows no current? *Ans.* 5,000 ohms.

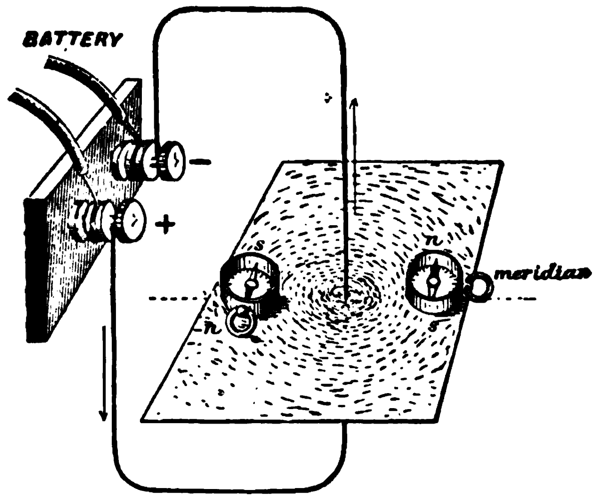
CHAPTER VI.

ELECTRO-MAGNETISM.

648. The Current's Lines of Magnetic Force.—If a wire, carrying a current of electricity, be passed through a sheet of paper, as indicated in Fig. 359, and if iron filings be sprinkled upon the paper, they will arrange themselves so as to form circles around the wire. If then a short magnetic needle be moved about the wire, it will tend to place itself tangentially to the circle passing through its centre. If the direction of the current be reversed, the needle will turn through 180°. The circles of the filings show the paths of magnetic lines of force, which owe their existence to the electrical current, just as the filings in Fig. 335 showed the paths of the lines of force of a magnet. In order to give a direction to these circular lines we must consider in what direction an isolated north magnetic pole would move. In the diagram this

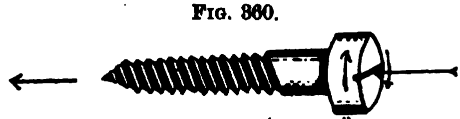
would evidently be contrary to the motion of the hands of a clock. In general, to remember the directions which these lines will have,

FIG. 359.



Maxwell makes use of the thrust and turn of an ordinary screw (Fig. 360). Suppose the current to flow along the axis of the screw, from the head to the point when it is being screwed into anything, and *vice versa* when it is being removed—*i.e.*, the direction of the current is the same as the direction of propagation of the screw—then the direction of the circular lines of force is the same as the motion of the circumference of the head of the screw when it is screwed in or out.

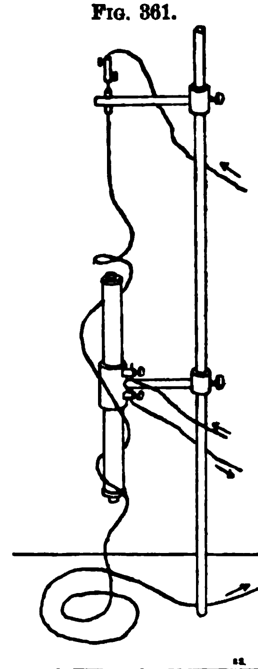
FIG. 360.



649. Effect of a Current on a Magnet.— The experiment in the preceding article shows that there is a connection between electricity and magnetism. In 1819, Oerstedt showed that a magnet tends to set itself at right angles to a wire carrying an electric current. He found, further, that the way in which the north end of the needle turns, whether to the right or left of its normal position, depends upon the position of the wire that carries the current—whether it is above or below the needle—and upon the direction in which the current flows through the wire. The position which a magnet will tend to take, when under

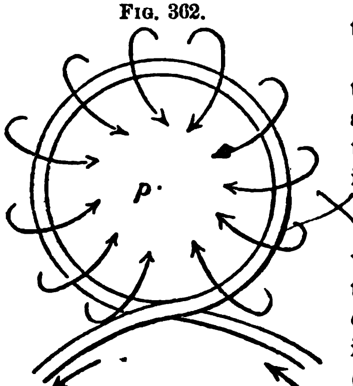
the influence of a current, can be easily determined by knowing the direction of the lines of force of both current and magnet, and by considering that the magnet will move in such a direction as to tend to bring its lines of force into the same path and direction as the lines of force of the current. Sometimes the student forgets the direction of the current's lines. In such a case let him remember that if the current flows from South to North and Over the needle, the north end of the needle will be turned toward the West, the combination being remembered by the initial letters, SNOW.

FIG. 361.



If we suppose the magnet to be fixed, and the conductor carrying the current to be movable, then the conductor will move because of the strife toward parallelism of their lines of force. Lodge illustrates this by a beautiful experiment. Send a strong current through a vertically suspended gold thread (such as is used upon military garments). Alongside the thread place, vertically, an electro-magnet (Fig. 361). Upon exciting the magnet the thread will wind itself around the magnet. Reverse the current and it will unwind and then rewind itself in the opposite direction. Complete parallelism of the lines of force is, of course, impossible, but the experiment well illustrates the tendency.

FIG. 362.



The movement of a needle under the influence of a current furnishes a convenient means of determining the direction in which the current is flowing.

650. Solenoids.—If a wire which carries a current be bent into a circle, all the lines of force will emerge from one side of an imaginary disc bounded by the loop (Fig. 362) and bending around the wire will enter the opposite side. The loop, because of the current, will be magnetically equivalent

to a disc magnet having north polarity on one side and south polarity on the other. In the diagram an isolated north pole placed above the surface of the page would be attracted toward *p*, as though it were a south magnetic pole. If the wire is coiled into the shape indicated in Fig. 363, it is termed a *solenoid* or *helix*. Upon passing a current, the lines of force, from their mutual action, take the paths indicated in the figure. A solenoid, when traversed by a current, has the same magnetic effect as a bar magnet whose axis coincides with the axis of the solenoid. Solenoids exhibit all the properties of magnets—attract pieces of soft iron, attract and repel magnets or other solenoids, and, if suspended by non-restraining quicksilver contacts, as in Fig. 364, will turn into the earth's magnetic meridian.

FIG. 363.

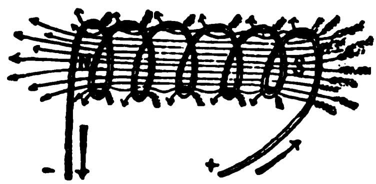
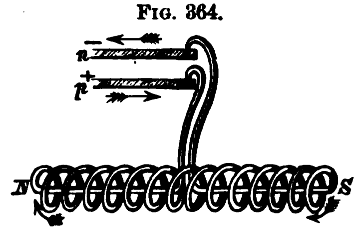


FIG. 364.



651. Ampere's Theory of Magnetism.—Because of the like actions exerted by solenoids and magnets, Ampere concluded that the permanent magnetism of steel owed itself to circular molecular currents of electricity, as shown in Fig. 365. He showed that the resultant of these many molecular currents was equivalent to surface solenoidal currents, as indicated in Fig. 366. In the interior of the magnet currents on contiguous molecules are running in opposite directions, and accordingly neutralize each other's magnetic effects. Half of the currents on the surface molecules are not neutralized, and the combined effect is the same as a surface solenoidal current.

FIG. 365.

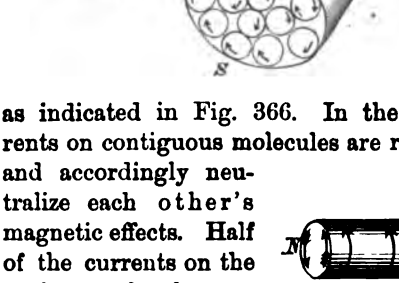


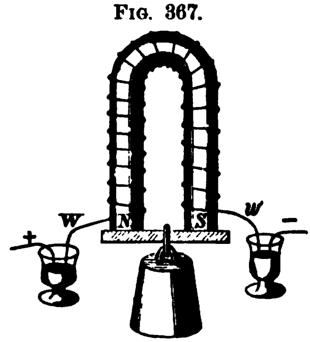
FIG. 366.

It should be remembered that looking at the north end of a magnet, end on, the amperian currents run counter-clockwise.

652. Electro-magnets.—We have seen (Art. 611) that when a piece of iron is placed in a magnetic field, it becomes an induced magnet and it adds lines of force to the field. Now, if an iron core be inserted into a solenoid, it becomes a magnet under the influence of the solenoid's field, and, because of its much greater permeability than air (Art. 612), adds many lines of force to the field. Such a combination is termed an *electro-magnet.* An electromagnet differs from an ordinary one in that the instant the exciting current is removed the electro-magnet loses its magnetism.

The intensity of the field which a given solenoid can produce is limited only by the strength of the current traversing it. If the current strength is doubled, the strength of the field is doubled. The iron core, upon being inserted, multiplies the strength of the field by a certain factor (the permeability of the core, see Art. 612). Now, if the permeability of iron were constant, there would be scarcely any limit to the strength which could be given to an electromagnet. But as the iron reaches the point of saturation its permeability decreases toward unity. As it is, electro-magnets can be made many times more powerful than permanent magnets.

FIG. 367.



A common form of electro-magnet is schematically shown in Fig. 367. The solenoid with its core is bent into the form of a horseshoe. An actual magnet would be wound with many more turns of wire, which must, of course, be insulated from the core. Upon passing a current, a heavy weight can be suspended, and this will detach itself as soon as the current is discontinued.

653. Magneto-motive Force.—In the practical construction of electro-magnetic apparatus it is often desirable to obtain a maximum number of lines of force in a given region. This region is to be occupied by some movable *armature* or object, to be subjected to the field's influences. Let us consider how this can be obtained when this region is the space between the poles of an electro-magnet. Evidently by increasing the number of lines generated by the current and adding as many lines as possible to these by proper selection of materials and shape of the electro-magnet.

The number of lines originally produced will increase with the current strength which flows through the solenoid or coil. It will also increase with the number of the loops of the wire in the coil—for each loop will add the same number of lines to those already traversing the axis of the coil. Accordingly we must employ as strong a current and as many turns of wire as possible. Remembering that the magnetic permeability of a substance is the same as its conductivity toward lines of force, it is desirable that all the space which is traversed by lines of force, except the portion which is to be employed for the movement of the object to be subjected to the field's influence, should be occupied by a substance of maximum magnetic permeability, *i.e.*, by the best soft iron.

Now we can obtain a law for the flow of lines of magnetic force exactly like Ohm's law (Art. 641) for current flow. Call the source of the lines the *magneto-motive force* (*M. M. F.*); call the reciprocal of the conductivity the *magnetic resistance* (*R*); then the magnetic flux or number of lines which pass through the axis of the coil

$$N = \frac{M.\,M.\,F.}{R}.$$

Evidently the *M. M. F.* is a function of the current strength (c) and the number of loops (n) made by the coil wire. It can be shown mathematically that if c is expressed in amperes, and N is to be obtained in absolute units, the *M. M. F.* $= \frac{4\,\pi}{10}\,n\,c$. The magnetic resistance is subject to the same law as electrical resistance (Art. 639). Increase the length l of the path to be travelled by the lines and the resistance is increased. It is decreased by increasing the cross-section q, and decreases with increase of magnetic permeability μ, so that the resistance

$$R = \frac{l}{\mu\,q}.$$

Introducing these values in the equation for the flux, we obtain

$$N = \frac{4\,\pi\,n\,c}{10\,\dfrac{l}{\mu\,q}}.$$

This formula is of great importance and has been of great service in the designing of efficient electro-magnetic machinery, *e.g*, dynamos and motors. To understand the application of it, let us refer to Fig. 367. The region where maximum flux is desired is the bottom of the horseshoe where the armature and the weight attached to it are suspended. The flux will be increased by increasing the current (c), the number of turns of wire (n), the cross-sections (q) of the core, the armature and the air gaps between the

armature and the poles, by increasing the permeability (μ) of the core and armature, and by decreasing the average lengths (l) of the core, the armature, and the air gaps.

When it is considered that the permeability of iron is much greater than that of air, it will be seen that the force of attraction would be greatly lessened if a piece of the iron core were removed from the top. The force exerted by a horseshoe electro-magnet is much greater than that exerted by two parallel straight electro-magnets corresponding to the two legs of the horseshoe.

654. The Morse Telegraph System.—The fact that an electro-magnet loses its magnetism as soon as the exciting current is discontinued, was made use of by Professor Morse in the construction of his system of electric telegraph. In this system an operator at one station can, by making and breaking a current of electricity which traverses a wire to a second station, produce or destroy, at will, the magnetism of an electro-magnet in the second station. This electro-magnet is a part of an instrument called a *register*, which will be described later. According as the magnet is excited for a longer or shorter interval, the register marks upon a moving band of paper a series of dashes or dots. These may be combined so as to serve as an alphabet.

The Morse circuit has four elements: A *battery* to produce a current; a *key* to manipulate the current; a *register* or *sounder* to record the current thus manipulated; and a *line* to convey the current.

The battery generally employed is a modification of the Daniell's type, called the Gravity Battery (Art. 635). Dynamos are, however, rapidly supplanting them in the large telegraph systems.

The *key* for manipulating the current consists of a *lever*, B (Fig. 368), and *anvil*, C, both of brass, and insulated from each

FIG. 368.

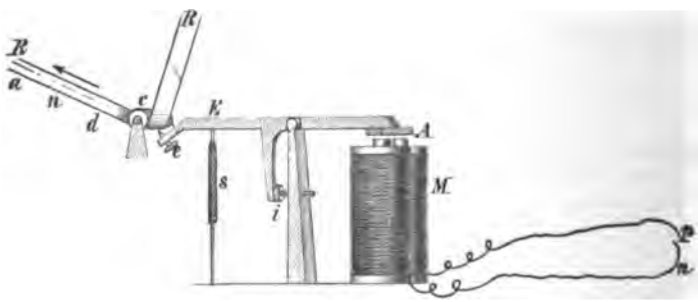
other. The anvil is connected to one terminal of the line, say from the battery, and B to the other terminal, where it leaves for the receiving station. The end of B is depressed by the finger of

the operator on the insulating button *F*, and is raised by the spring *E*, when the pressure is removed. The former movement closes the circuit, the latter opens it, and by a succession of these the message is sent. When the key is not in use, the brass bar *K*, hinged to the base of *B*, is pressed into contact with *C*. This closes the circuit so that other operators on the line may have a continuous circuit when they desire to send a message. When not in use, the line is traversed by a current.

The *register* for recording the message on paper is constructed as follows:

The lever *E* is furnished with a *style* *e* (Fig. 369), directly over which is a groove on the surface of a solid brass roller *c*. Between *c* and *e* is a long paper ribbon *R R*. Attached to *E* is a soft iron armature *A*, placed *above* the magnet *M*, and furnished with a

FIG. 369.



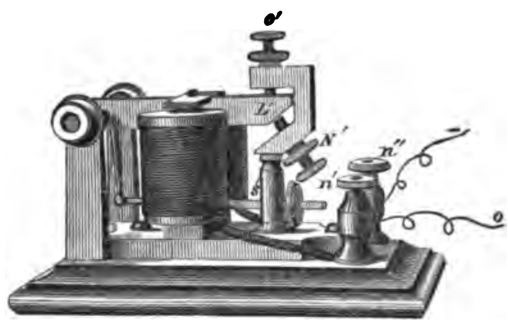
spring *s* to raise it as far as the screw *i* allows when it is not attracted by *M*. When the circuit is closed, *A* is attracted and *e* rises and forces the paper into the groove, producing a slight elevation on its upper surface. The ribbon is pulled along at a uniform rate in the direction of the arrow by clockwork (not shown in the figure), so that when the circuit remains closed for a little time, a *dash* is marked on the paper by *e*; when it is closed and instantly opened, the result is a *dot*—or rather a *very short dash*. *Spaces* are left between these whenever the circuit is opened. Combinations of these *dots*, *dashes*, and *spaces*, all carefully regulated in length, compose the letters of the alphabet. Spaces are also left between the letters, and longer ones between words.

By lengthening the circuit wire, it is evident that the person who sends the message at *n p*, and the one who receives it at *E*, may be miles apart, and the transmission will be almost instantaneous, owing to the rapid passage of the current.

It has been found that the ear is sufficiently accurate to allow of the dispensing with the register, as used by Morse. Instead of

it a *sounder* is employed. In this the end of the lever L' (Fig. 370), instead of being furnished with a style, is made to strike against

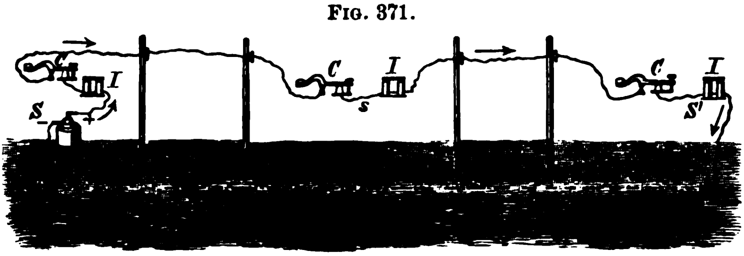
FIG. 370.



the two screws, N', O'. The downward *click* is a little louder than the upward one, and so the beginning and end of each *dot* or *dash* are distinguished from each other. Many operators learn from the first to *read by the ear*, and have never used a register.

For a *line* it was at first supposed that a complete metallic circuit was necessary, hence a return wire was employed. But this was rejected when it was found that the earth could be used as a part of the circuit, as shown in Fig. 371, in which the dotted line and arrow beneath the surface are not intended to convey the idea that a current actually flows from one earth-plate to the other, but

FIG. 371.

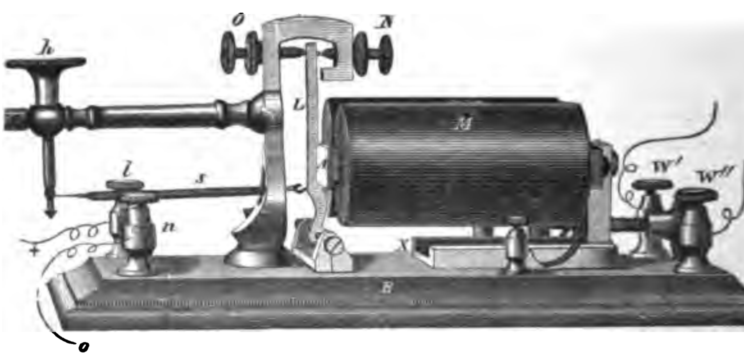


that a complete circuit is formed, the earth acting the part of an infinite reservoir of electricity. S and S' are the terminal stations, and s is one of the way stations which may occur anywhere along the line. At every station both a key, C, and sounder, I, are introduced into the circuit, so that messages can be both sent and received.

655. The Relay.—When a telegraph line is very long, its resistance is high and the leakage, because of insufficient insula-

tion, is great. Hence a current sufficiently strong to satisfactorily operate a register or sounder cannot be economically sent through it. Accordingly use is made of a *relay*. In this instrument (Fig. 372) the line current entering at W' and leaving at W'' excites the electro-magnet M. This attracts the armature A of the delicately adjusted lever L. The adjustment is obtained by regulating the tension exerted by the spiral spring s. During the passage of a current along the line, the lever L plays lightly to and fro, but with insufficient strength to act as a register or sounder. It can, however, be made to act as a *key* for a *separate local circuit* in the receiving office. One terminal of this local circuit, which contains a sounder and battery, is connected by the binding-post l with the

FIG. 372.



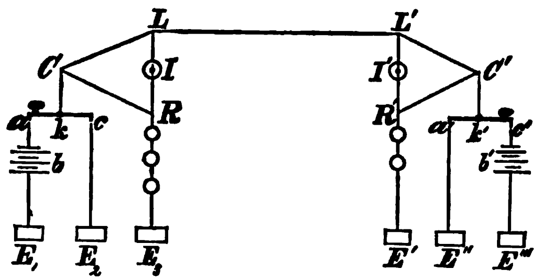
lever L. The other, through n, is connected with the screw N. When the distant operator closes his key the armature A causes the lever L to close the local circuit at N. When the distant operator opens his key, the spring s opens the local circuit. Thus the moving lever of a relay acts as a key for a local circuit.

Evidently the relay may be used for *repeating* a message on another long circuit.

656. Duplex Telegraphy.—In the Morse system just described evidently but one message can traverse the wire at the same time. If two could simultaneously traverse it, the earning capacity of the line would be doubled. This feat can be accomplished, and is termed *duplex* telegraphy. A simple duplex system, employing the principle of Wheatstone's bridge (Art. 646), is shown in Fig. 373, which represents two stations connected by the line wire $L\ L'$. $C\ L\ R$ is a Wheatstone bridge, modified to suit the conditions of the case, I the sounder, R resistance coils, k a key working upon the centre and having forward and back con-

tacts at a and c, b the battery, and E the earth connections. The same letters, accented, represent like parts at the second station.

FIG. 373.



When not in use the keys make back contact by the action of a spring. The *ratio* of the resistance $C\ R$ and $R\ E_3$ is made equal to that of $C\ L$ and the line wire $L\ L'$, *including the back contact earth connection* at the second station. When thus balanced any current arriving at C, which, dividing, passes through $C\ L\ L'$ and $C\ R\ E_3$, will maintain the points L and R at the same potential.

If now, a' being closed, a be closed, a current will flow through a and k to C, where it will divide, one part going to earth through R and E_3, and the other through $L\ L'$. As the potentials were made equal at L and R, no current will pass through the indicator I; that part of the current which flows through $L\ L'$ divides at L', part going through $C'\ k'\ a'$ to E'', and part through I' (giving signal) and R' to E'. Thus the closing of a gives a signal at I' but none at I.

If now the second operator should close his key while a was closed, a current from b' would flow through c' and k' to C', where it would divide, part going to earth through R' and E' (joining the current already flowing through from $L\ L'$), and part would flow to L' and oppose the current from the other station; this opposing current will have the same effect as increased resistance in the line wire $L\ L'$, and hence the balance $C\ L\ R$ will be disturbed, the potential of L rising above that of R, and resulting in a current from L through I to R, giving a signal at I. Thus the register at each station will respond to the key of the other, and only to that, whether one or both operators be signalling.

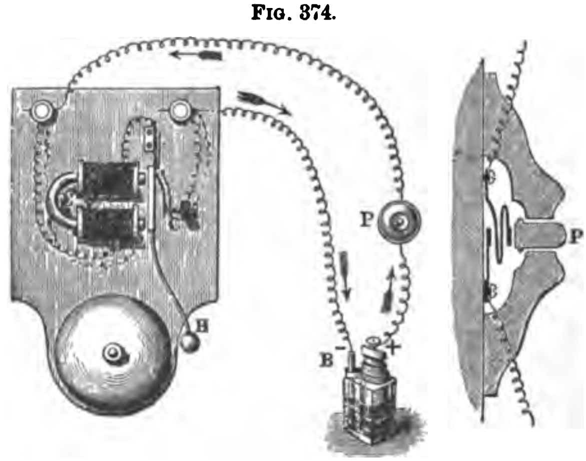
The above explanation of the principle of this particular mode of sending simultaneous messages in opposite directions on a single wire, does not pretend to describe the *actual* arrangement of wires or earths in use. For a full description of the various modes of duplex and quadruplex telegraphy the student is referred to works on practical telegraphy.

657. Atlantic Telegraph Cable.—This cable stretches a distance of 3,500 miles, and from the nature of the case is a continuous wire, so that it cannot be advantageously worked by the Morse apparatus. The indicator employed is a sensitive galvanometer needle, which is made to oscillate on opposite sides of the zero point by the passage through it of currents in opposite directions. But to reverse the direction of the current throughout the whole length of the cable is a slow process. *For the cable is an immense Leyden jar*, the surface of the copper wire (amounting to 425,000 sq. feet) answering to the *inner* coating, the water of the ocean to the *outer*, and the gutta-percha between the two to the *glass* of an ordinary jar. A current passing into it is therefore detained by electricity of the contrary kind induced in the water, and no effect will be produced at the farther end until it is *charged*.

This very circumstance, at first considered a misfortune, is now taken advantage of in a very simple and ingenious manner to facilitate the transmission of signals. The current is allowed to pass into the cable till it is *charged*—then, *without breaking the circuit*, by depressing a key for an instant, a connection is made between it and a wire running out into the sea; that is, between the inner and outer coatings. *This partially discharges it*, and the needle at the other end is deflected. When the key is raised the discharge ceases, the current flows on as before, and the needle is deflected in the opposite direction.

658. Electric Bells.—The ordinary electric house bell consists of an electro-magnet, which moves a hammer backward and

FIG. 374.

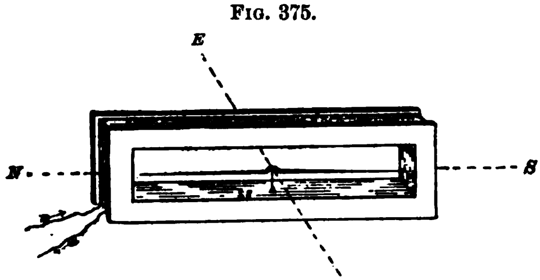


forward by alternately attracting and releasing it, so that it beats against a bell. The arrangements of the instrument are shown in

Fig. 374. A current from a battery (usually of the Leclanche pattern), after traversing the electro-magnet E, enters a spring attached to the armature and bell-hammer. It leaves the spring by an adjustable screw, C, and returns to the battery. When it flows it excites the magnet which attracts the armature and causes the hammer to hit the bell. In moving toward the magnet the contact at C has been broken, and the magnet losing its magnetism allows the armature to spring back so that the contact is renewed. This operation is repeated, the current repeatedly making and breaking itself. One of the wires from the battery to the bell is cut at the point P, and a push button is inserted. This is shown in section to the right. An insulating knob, P, when pressed, brings a spiral spring, which is connected with one end of the cut wire, into contact with the other end. The circuit being closed thus, the bell commences to ring.

659. Galvanometers.—These instruments are employed in the laboratory for the determination of nearly all electrical magni-

FIG. 375.



tudes. They serve to detect the presence of electrical currents and to determine their strengths and directions. The principle of their action is electro-magnetic. Suppose a magnetic needle (Fig. 375), free to move about a pivot, to lie in the direction of the earth's magnetic meridian. Suppose further, that it be surrounded by a coil of wire, whose windings are parallel to the axis of the needle. If, now, an electrical current be sent through the coil, it will develop magnetic polarity in the coil so that, *e.g.*, its east side will be equivalent to a north pole and its west side to a south pole. The needle will, under this influence, tend to place itself in an east and west direction. It will not quite attain this direction, for it is influenced by the earth's magnetism at the same time, and this tends to keep it in the meridian. Upon reversing the direction of the current, the polarities of the sides of the coil become reversed, and the needle turns so that its poles project from opposite sides

of the coil. The side toward which the north end of the needle turns determines the direction of the current in a given galvanometer. The angle through which the needle is deflected determines the strength of the current flowing.

TANGENT GALVANOMETERS.—If the wire of a galvanometer be wound on the circumference of a ring, whose diameter is at least twelve times the length of the needle at its centre, the strengths of currents causing different deflections will be proportional to the *tangents* of the corresponding angles of deflection. Such an instrument is called a *tangent galvanometer*. The reason for having a large diameter for the coil is that those of its lines of force, which are cut by the short needle in its excursions, are then straight and perpendicular to the earth's lines. The magnet's pole is thus moved under the influence of two forces, which act continuously at right angles to each other. The law of the tangents then follows.

REFLECTING GALVANOMETERS.—In refined laboratory measurements the determination of a needle's deflection, by observing the movement of a pointer over a divided scale, is inaccurate and inconvenient. Instead, a small mirror is attached to the magnet and the deflections are measured by the different divisions of a stationary divided scale, which are reflected from the mirror into a stationary telescope. The arrangement is shown in Fig. 319.

A method, much used in England, is to have the mirror reflect a ray of light from a small hole in an opaque chimney of a lamp upon a stationary scale. The method is very inconvenient, as it requires the observations to be made in a darkened room. The accuracy to be obtained is not as great as by means of a telescope and scale.

BALLISTIC GALVANOMETERS.—In many determinations it is required to measure currents which last but for an instant, or to measure *quantities* of electricity. The difficulties connected with these determinations are much lessened if the time required by the galvanometer needle to make a single oscillation be very great, as compared with the time occupied by the electricity in passing. Thus galvanometers whose needles have periods of from five to twenty-five seconds are used, and are called *ballistic galvanometers*.

DIFFERENTIAL GALVANOMETERS.—These instruments are supplied with two sets of coils, which are so placed that they will produce the same electro-magnetic effect upon the single needle, providing they be traversed by currents of the same strength and direction. By means of this instrument a current in one coil may be brought to a given strength by being made to neutralize the effect upon the needle from another current, which is of constant (the required) strength and passes through the other coil in an opposite direction.

CHAPTER VII.

ELECTRO-DYNAMICS.

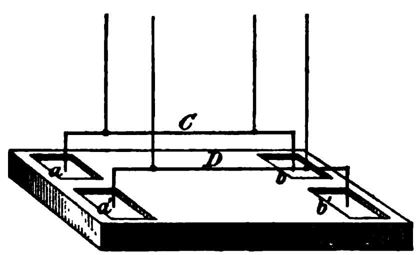
660. Movement of Conductors Carrying Currents.—In the preceding chapter it has been shown that a conductor carrying an electrical current, and placed in the vicinity of a magnet, tends to move the magnet, so that the lines of force from each may become parallel, or, if the magnet be stationary, the conductor strives to move, to attain the same end. As might be expected, two neighboring conductors, while traversed by currents, tend to move so as to render their lines of force parallel.

Without any knowledge of the existence or properties of lines of force, Ampere, in 1821, arrived, by experiment, at the following laws, which could easily have been predicted by such a knowledge.

661. Parallel Currents.—

1. If galvanic currents flow through parallel wires in the *same direction*, they *attract* each other; if in *opposite directions*, they *repel* each other. These effects are shown by suspending wires, bent as in Fig. 376, so that their lower ends may dip into four separate mercury cups, a, b, a', b', by means of which connection between the wires C and D and the battery may be readily made. The suspending threads should be two or three feet long, and the mercury cups should be large enough to allow considerable lateral movement of the wires. If simultaneous currents be sent through the two wires C and D, in the same direction, the wires will move toward each other; if currents be sent through the wires in opposite directions at the same time, they will separate more widely.

FIG. 376.



Hence, when a current flows through a loose and flexible helix, each turn of the coil attracts the next, since the current moves in the same direction through them all. In this way a spiral suspended above a cup of mercury, so as to just dip into the fluid, will vibrate up and down as long as a current is supplied. The weight of the helix causes its extremity to dip into the mercury below it; this closes the circuit, the current flows through it, the spirals attract

each other, and lift the end out of the mercury; this breaks the circuit, and it falls again, and thus the movement is continued.

2. If currents flow through two wires near each other, which are free to change their directions, the wires tend to become parallel to each other, with the currents flowing in the same direction. Thus, two circular wires, free to revolve about vertical axes, when currents flow through them, place themselves by mutual attractions in parallel planes, as in Fig. 377, or in the same plane, as in

FIG. 377.

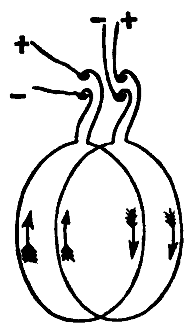


FIG. 378.

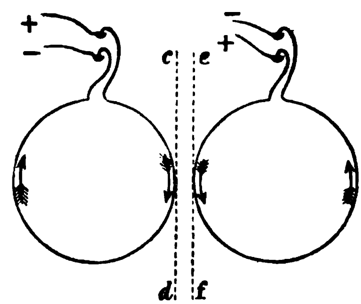


Fig. 378. In the latter case, we must consider the parts of the two circuits which are nearest to each other as small portions of the dotted straight lines, *c d* and *e f*.

It appears, therefore, that *galvanic currents, by mutual attractions and repulsions, tend to place themselves parallel to each other in such a manner that the flow is in the same direction.*

The force exerted between two parallel portions of circuits is proportional to the product of the current strengths, to the length of the portions, and inversely proportional to the distance between them. The force exerted by each current acts in a direction perpendicular to the direction of the current.

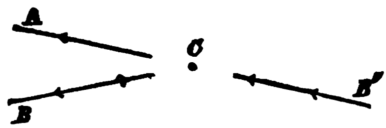
662. Currents not Parallel.—*Currents, both of which flow toward a common point, or both of which flow away from a common point, attract each other.*

If one of two currents flows toward, and the other away from a common point, the two currents repel each other.

These cases are evident deductions from the preceding paragraph. Suppose the two currents (Fig. 379) to flow in *A* and *B* as though they came from *C*, then the tendency of the wires *A* and *B* is towards parallelism, and as we suppose the currents to flow from the direction *C*, the wires must tend to move toward each

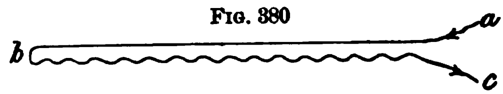
other in order to become parallel. The same effect would be produced if the currents in A and B were to flow towards C. But if the current in A flows *from* the direction C, and that in B *towards* the point C, then the tendency of the wires to become parallel, with the currents flowing in the same direction, causes B to revolve about C as a centre till it reaches the position B', and then the condition that the currents shall flow in the same direction will be fulfilled. It is not necessary that we should regard A and B as lying in the same plane.

FIG. 379.



A sinuous current produces the same effect as a straight current having the same general direction and length. If a conductor, having one portion sinuous and the other straight, be bent as in Fig. 380, so that the current may flow from a to b through the

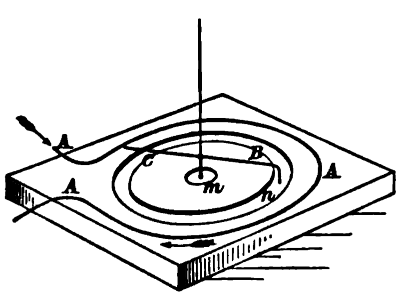
FIG. 380



straight part, and from b to c through the sinuous part, the two portions of the current thus flowing close together in opposite directions, the joint electro-dynamic effect upon a movable conductor parallel to $a\ b$ will be inappreciable.

663. Continuous Rotation Produced by Mutual Action of Currents.—Suppose a continuous current to flow through a wire A, as indicated in Fig. 381, and that a wire B, so bent as to dip into the mercury cup m at one end, and into the annular mercury trough n at the other, be suspended at the middle, a counterpoise, C, keeping it balanced.

FIG. 381.



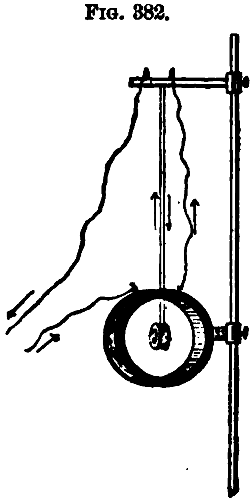
If, now, a current be made to flow from the cup m, through B, and thence out again by means of the mercury contact in n, the wire B will rotate in a direction opposite to that of the current in A; for the current in B, and that in the part of A to the right of n, are both flowing towards n and hence attract, while the current in B and that part

of the current in A immediately to the left of n are flowing in directions to cause repulsion.

A beautiful experiment, illustrating continuous rotation, is to place a round, shallow dish, containing mercury, on the pole of a vertical, straight electro-magnet. Excite the magnet and dip the terminals of a circuit, carrying a strong current, into the mercury at the centre and side of the dish respectively. A portion of the mercury carries the current from the centre to the edge of the dish. In doing so it is made to rotate by the action of the lines of force from the magnet. As soon as it has rotated a new portion of the mercury is made to carry the current. This, in turn, gives way to another portion, and the whole body of mercury is soon set into rapid rotation. Centrifugal force, resulting from the rotation, causes the mercury to heap up around the edges of the dish, and to be depressed at the centre.

664. Electro-dynamometer.—This instrument, invented by Weber, is used for measuring the strengths of electrical currents. Its action depends upon the electro-dynamic attractions discussed in Art. 661. The principles of its construction are shown in the crude apparatus represented in Fig. 382. This consists of a fixed hollow coil of wire, in the centre of which is suspended another smaller coil. The suspension is made by means of two fine parallel wires, placed one or two millimetres from each other. The upper ends of these wires are connected to two insulated binding-posts, and the lower ends are connected with the terminals of the suspended coil. The suspension is so arranged that, when no current is passing through the dynamometer, the planes of the two coils are perpendicular to each other. If, now, a current of electricity be sent through the apparatus (in the following order through the external coil, down one suspension wire, through the inner coil and up the other suspension wire), the suspended coil will turn and strive to cause a parallelism of the planes and currents of both coils. The turning force of the currents is resisted by an increasing force exerted by the twisted wire suspension. With a certain current the coil will be deflected a certain amount—*i.e.*, until the two opposing forces are equal.

FIG. 382.



With a stronger current the deflection will be greater. Thus the magnitude of the deflection can serve as a measure of the current strength.

A peculiarity of the electro-dynamometer is that it serves to measure alternating currents, *i.e.*, those which change their direction, perhaps, several thousand times per minute, equally as well as continuous currents. A change in the direction of flow of the main circuit changes the direction in *both* coils. This does not alter the direction of the deflection.

CHAPTER VIII.

ELECTRO-MAGNETIC INDUCTION.

665. Currents of Electricity Produced by Induction.—It has been shown that when a current of electricity flows through a conductor the air or other dielectric which surrounds the conductor is traversed by lines of force. The presence of these lines indicates that the dielectric is under some sort of a strain. To produce this strain energy must have been expended by the current when it commenced to flow. During the short time that the strain is being produced there is an opposition to the exciting current, which is equivalent to a current in an opposite direction. Now, it is reasonable to suppose that, if lines of force or a magnetic field be produced by some agency around a closed circuit which is primarily traversed by no current, a current will be produced in this circuit. The direction will be opposite to that which would be necessary to create the field, and will last only for the time necessary to produce the strain. Furthermore, upon destroying the field it is reasonable to suppose that the energy which it represents will appear as a current in the same direction as one which could produce the field. These suppositions are substantiated by experiment, as was first shown by Faraday. The currents are called *induced currents* (not to be confounded with induced electrostatic charges), and those currents whose directions are the same as a current which could produce the field are termed *direct currents*, while those in an opposite direction are called *inverse* currents.

666. Methods of Producing the Inducing Field.—Inasmuch as induced currents are produced by any variation in the strength of the field around the conductor which carries them, they can be produced either by varying the strength of the field

current or by moving the conductor into fields of various strengths. For the sake of clearness suppose that we are supplied with the apparatus represented in Fig. 383. c is the *primary coil* of wire which produces the field, and is traversed by a current from the battery. The *secondary coil*, in which induced currents are to be produced, is represented at d. Its terminals are connected with a galvanometer, which indicates the presence and direction of the induced currents. Now suppose that c be placed inside of d. Upon starting the current in c an inverse current will be induced in d, and upon stopping it a direct current will be induced. Permitting the current in c to flow, increasing or decreasing its strength will produce inverse or direct induced currents respectively. If the current strength in c be maintained constant, removing the coil c will produce a direct current, and replacing it an inverse induced current.

FIG. 383.

Induced currents may also be produced by magnets. Consider a magnet to be the equivalent of a solenoid traversed by a current (Art. 651). Dispensing with the battery we have the apparatus indicated in Fig. 384. An inverse current will be induced by the introduction of the magnet into the secondary coil, and a direct cur-

FIG. 384.

FIG. 385.

rent upon removing it. An inverse current may also be induced by strengthening the field of a magnet which is stationary within the secondary, by bringing a piece of iron near to it. In this case the iron becomes a magnet by induction, as shown in Fig. 385, and adds its lines of force to the field. Direct induced currents will follow the removal of the iron.

It is well to remark that, as motion is merely relative, it is immaterial whether a magnet be placed in a secondary coil or the latter be placed around the magnet.

The facts which have been mentioned may be summed up in a single law:

Inverse induced currents always result from an Increase in the number of lines of force which pass through the circuit, and Direct induced currents always result from a Decrease in the number of these lines.

667. Lenz's Law.—*If two conductors, A and B, in one of which, A, a current is flowing, be made to change their relative positions, then a current will be induced in B in a direction which will cause a mutual action in the two conductors tending to oppose their motion.* Thus, if A and B be brought nearer together an inverse current will flow in B, and currents flowing in opposite directions repel each other; and if A and B be caused to move apart, then a direct secondary current will flow in B, and currents flowing in the same directions attract each other. This statement of the results of experiments will aid the memory in regard to the directions of the primary or secondary currents.

668. Self-Induction.—Whenever a current is started in a coil of wire, lines of force are created which increase in number from zero to a maximum. Owing to the increase, they induce currents in the coil which are opposite to the direction of the original current. Upon stopping the original current the lines of force decrease in number and thus induce a direct current in the coil. The induction in such a case is termed *self-induction*, and the currents are termed *extra* or *self-induced* currents.

The existence of self-induced currents may be demonstrated by the Wheatstone bridge combination (Art. 646). Let three of the arms of the bridge be made up of resistances without self-induction (Art. 645), the fourth arm consisting of an ordinary unifilar coil. For the purpose of increasing the self induction of this fourth arm, insert a piece of soft iron in the coil. Obtain a balance in the bridge by employing a constant current. When a balance has been obtained the galvanometer will show no deflection. If the

current be now stopped, the current induced in the fourth arm will cause a deflection of the galvanometer needle.

669. Coefficients of Mutual and Self-Induction.—It can be proved mathematically that the *electro-motive force induced in a closed circuit is equal to the rate of variation of the number of lines of force which pass through it.*

If in a short interval of time dt, the number of lines of force N increases a small amount dN, then the electro-motive force

$$E = -\frac{dN}{dt}$$

will be induced in the circuit which surrounds these lines. In case two coils, a primary and secondary, be fixed in position, and the strength of the current in the primary be increased by an amount dc in the short time dt, then the electro-motive force

$$E = -M\frac{dc}{dt}$$

will be induced in the secondary during that time. M is a constant which is called the *coefficient of mutual induction* between the two coils. Its value depends upon the shape and number of windings of wire around the respective coils and their relative positions. It is numerically equal to the number of absolute lines of force which would be sent through either coil when an absolute unit current of electricity was sent through the other coil. It makes no difference which coil be chosen as a primary in determining M.

If it be supposed that the two coils be made to coincide, *i.e.*, that there be but one coil, then the electro-motive force of self-induction

$$E = -L\frac{dc}{dt}.$$

The constant L is called the *coefficient of self-induction*, and is equal to the number of absolute lines of force which a coil would send through itself if it were traversed by an absolute unit of current.

670. Induced Currents from the Earth.—If a coil (whose terminals are connected with a sensitive galvanometer) be placed so that its axis is parallel with the axis of a dipping-needle (Art. 621), it will be pierced by a maximum number of the earth's lines of magnetic force. If it be now turned through 90° around an axis perpendicular to its own axis, the number of the lines piercing it will decrease to zero, and the galvanometer will indicate that a current is being induced by the rotation.

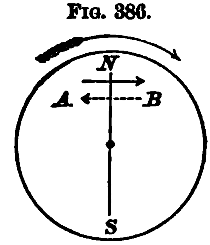
Continuous variations in the strength of the earth's magnetism sometimes induce currents of considerable strengths in long telegraphic circuits. Such currents are known as *earth currents.*

671. Arago's Rotations.—In 1824 Arago observed that the oscillations of a magnetic needle were reduced in number by suspending a copper plate above it. This observed phenomenon soon led him to the discovery that if a horizontal copper disc be made to rotate rapidly, a magnetic needle suspended above it would rotate also. This effect may also be produced with other metals though in less degree.

If a disc of copper be set spinning on an axis, between the poles of a powerful electro-magnet whose circuit is broken, the axis of the disc being parallel to the lines of force, the rotation continues with slight loss of velocity for a long time; but if the circuit be suddenly closed the rotation is at once checked, or possibly stopped. If such a disc be kept in rapid rotation by a suitable band and pulley, after the circuit is closed, the disc will be heated by the action of the magnet.

These effects were explained by Faraday as being due to currents induced in the mass of metal. Thus let a needle, *N S* (Fig. 386), be suspended above a metal disc *A B*. The magnetic currents flow around the needle as indicated in the figure, the currents below the needle from right to left as shown by the dotted arrow, and those above from left to right, as shown by the full arrow. Now suppose the disc to be rotated in the direction from *A* to *B*; the portions of the currents around *N S* which are nearest to the disc will induce in that part of the disc towards *A* currents whose directions are such as to resist the motion of the disc, according to Lenz's law (Art. 667), that is to say, currents will flow in the disc from left to right; while in that part of the disc towards *B*, which is moving away from *N*, the induced currents are from right to left, and so resist the motion of *B away* from *N*.

FIG. 386.

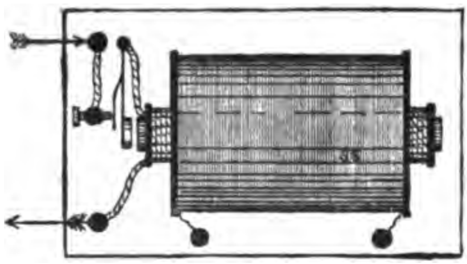


If the needle had been moved, the disc remaining fixed, the same analysis of the motion might be made, and we should find that the disc would resist the motion of the needle. A copper collar or frame is sometimes used to coil the galvanometer wire upon, in order to reduce or *damp* the oscillations of the needle, and bring it more quickly to rest.

672. Induction Coils.—These instruments serve to transform currents of low E. M. F. into alternating currents of high E M F. Their forms and sizes are many, and only their principle need be mentioned here. A continuous current of low E. M. F. is passed through a primary coil made of a few turns of coarse insu-

lated copper wire (Fig. 387). The centre of the coil is filled with a core of soft iron wires. Before passing through the coil the current traverses some sort of a current-breaker, *e.g.*, the one shown in the cut acts upon the same principle as the breaker of the electric bell described in Art. 658. By means of this breaker the current in the primary is rapidly made and broken. Alternating currents are thus induced in a secondary surrounding coil, which is wound with many turns of very fine insulated wire. The E. M. F. of these induced currents is great because the coefficient of mutual induction is great. This is owing to the large number of turns of wire in the secondary and to the presence of the iron core. Both conspire to cause a large number of lines of force to pierce the circuit during the short interval required to make the circuit of the primary.

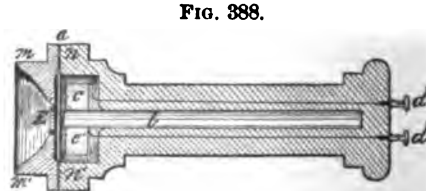
FIG. 387.



The function of the induction coil, as here given, is often reversed, in which case it becomes what is termed a *transformer*. Transformers are much used in the commercial distribution of rapidly alternating currents for lighting and other purposes. Alternating currents of high E. M. F. and low current strength are received from a main line into the finer wire coil of an induction coil. The thick wire coil of the transformer is connected with the customer's home circuit, and delivers to it currents of great strength but at low potential.

673. The Telephone.—This instrument for reproduction of sound at a distance by means of electric currents is shown in section in Fig. 388, in which $a\ a'$ is a disc or diaphragm of thin soft iron, the circumference of which is firmly clamped between the mouth guard $m\ m'$ and the case $n\ n'$, upon the centre of which the sound-waves from the mouth impinge, as at E, and communicate to it vibrations corresponding to the simple or composite sounds uttered. These vibrations of the disc cause a continual variation in the distance of the disc from

FIG. 388.

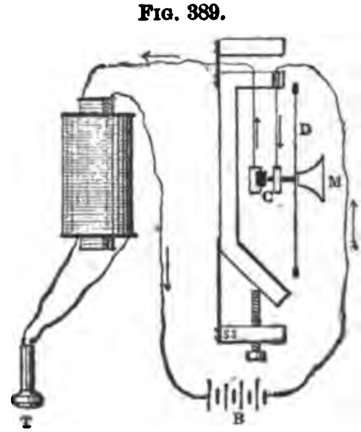


the end of a bar magnet, *b*. Around the end of the magnet nearest to the diaphragm *a a'* is a coil, *c c'*, of fine insulated copper wire, the ends of which are connected with binding-posts, *d d'*. From these posts are carried wires to another precisely similar instrument at the station with which communication is to be held. When a word is spoken into the instrument at *E*, the vibrations communicated to the disc *a a'* cause variations in the magnetic field of the bar *b*, and these variations induce electric currents which flow in the coil *c c'*, and thence through the connecting wires to the coil in the instrument held to the ear of the listener, and these currents in the last-named coil produce variations in the strength of the magnet of the receiving instrument, causing precisely the same vibrations in its diaphragm as were originally set up in the first. The vibrations of the diaphragm are transmitted through the air to the ear; and though no *sound* has been transmitted from one station to the other, the words spoken into one instrument are distinctly delivered by the other. The sound vibrations are the cause of electric currents, and these in turn finally produce sound vibrations again.

To such perfection of action have these instruments been brought, that not only can the spoken words be heard, but the peculiar characteristics of voice are so faithfully reproduced that by these the speaker may be recognized.

674. The Blake Transmitter.—The electro-motive forces generated by the moving diaphragm of the Bell telephone are not sufficiently large to produce satisfactory results on long lines. Therefore an instrument termed a *transmitter* is substituted for the telephone at the sending end of the line. A common and very satisfactory form of transmitter is one designed by Francis Blake. It is represented in Fig. 389, and its action depends upon the principle that the electrical resistance offered by a carbon contact varies greatly with the pressure exerted upon it.

FIG. 389.



The sound to be transmitted is received in the mouth-piece *M*, which causes it to set the diaphragm *D* into corresponding vibrations. Touching the rear of the diaphragm is a platinum or carbon point, which is attached to a piece of watch-spring, and which

is in connection with one terminal of a battery, B. (This battery is brought into circuit only as the transmitter is to be used.) The point forms a loose contact, C, with a carbon button, which is also mounted upon a spring. The current from the battery flows through this contact to the rest of the circuit. As the diaphragm vibrates it causes the point to exert correspondingly different pressures upon the button. The resistance of the circuit is thus varied, and this results in variations in the current which are the electrical counterparts of the sound vibrations.

A Bell receiver, placed in the same circuit with the battery and transmitter, will yield, besides the transmitted sound, a disagreeable "sizzling" noise. To obviate this an induction coil, I, is introduced. The varying current from the battery and transmitter is passed through the primary of a small induction coil, and the line wires, with their receivers included, are connected with the secondary coil. In this case the currents on the line flow in opposite directions to what they would, if connected directly with the transmitter circuit. This, however, is of no consequence.

The springs of the transmitter, which bear the carbon button and platinum point, are fastened to one piece of brass, but are insulated from each other. The amount of pressure at the contact is regulated by a screw, whose end hits the bent end of the brass holder. The holder is supported by pliable spring bands which are attached to the case of the transmitter. Although this method of adjustment is simple and appears crude, the delicacy of it is marvellous.

675. Dynamos.—These machines are for converting mechanical energy into electrical currents. A discussion of the principles of their construction is here out of place, and the student is referred to some one of the many technical treatises on the subject. The principle of their action may be described.

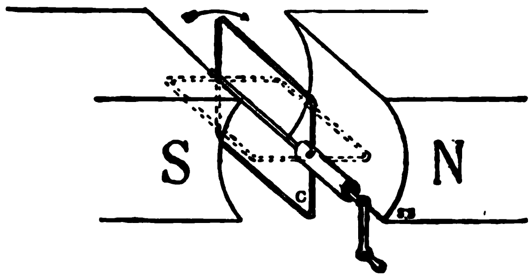
The dynamo has two essential parts—a movable* conductor, called an *armature*, and a magnet, in whose field the armature moves. The armature, by its motion, varies the number of the field lines which pass through it, and is therefore traversed by induced currents.

Fig. 390 (taken from S. P. Thompson's "Dynamo-Electric Machinery") represents an ideal dynamo in its simplest form. N and S are the poles of a field electro-magnet. The lines of force pass between the poles and pierce the looped conductor C, which forms the armature. The armature, in the position indicated by the con-

* In some machines the armature is stationary and the field magnets are movable.

tinuous lines, is pierced by a maximum number of the field lines; upon turning through 90°, coming then into the position indicated

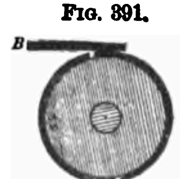
FIG. 390.



by the dotted lines, it is pierced by none of the lines. During the whole quarter revolution there has been a decrease in the number of lines passing through the loop. An induced current, flowing in a certain direction, has accompanied the movement. During another quarter revolution the number of penetrating lines will be on the increase, but they now pass through the loop in an opposite direction to what they did before, and hence the induced currents which result from the increase are in the *same direction*, referred to the conductor, as during the first quarter revolution. During the next two quarter revolutions the induced currents will flow in an opposite direction. Thus by continuous revolution the armature is traversed by currents which reverse their directions twice each revolution.

In order to lead the currents from the armature into a circuit where they can be used, and in order to rectify them, *i.e.*, cause them to flow in the same direction, use is made of a *commutator*. Fig. 391 represents a two-part commutator suited for our single-loop armature. It consists in an insulating cylinder, to be applied to the extremity of the axis of the armature. Upon it is slid a metal tube slit into two parts. To each part is connected one of the ends of the loop, as shown in Fig. 390. Against the commutator are pressed two spring *brushes*, *B B*, which are connected with the two terminals of the outside circuit respectively. The commutator revolves with the armature, but the brushes remain stationary. Both are so arranged that at the instant the plane of the loop of the armature passes through the vertical plane, the brushes will slide from one segment of the commutator to the other. At this instant the induced current reverses the direction of its flow, and the commutator, exchanging

FIG. 391.



the connections with the external circuit, causes the external current to flow in one direction.

The E. M. F. which could be obtained from such an ideal dynamo would be very small. To increase it, the total number of lines of force which are passed through or taken out of the circuit in a unit time must be increased. There are three ways in which this may be accomplished: the *speed* of revolution, the *number of loops* in the armature, or the strength of the *field* may be increased. It need not be considered here how this is carried out in practice.

The field magnets of a dynamo may be excited by currents from an external source, by the whole of the machine's armature current, or by only a portion of the armature current. The dynamos are then termed *separately excited*, *series*, or *shunt* machines respectively.

When the field is furnished by permanent magnets the machine is no longer termed a dynamo, but a *magneto-electrical generator*. Such machines are not a commercial success except in the very small sizes.

676. Electric Motors.—The function of these machines is the converse of that of dynamos. They are intended to transform electrical energy into motion. The dynamo of the previous article becomes an ideal motor by simply sending through it, from the external circuit, a current in an opposite direction. The commutator accomplishes that the lines of force, due to the current flowing in the armature, shall never become parallel to the field's lines. In striving to secure such parallelism the armature revolves upon its axis, and just as it is about to reach the goal the commutator reverses the direction of its lines, and it moves through another half revolution to be again frustrated in its attempts.

CHAPTER IX.

ELECTRO-CHEMISTRY AND ELECTRO-OPTICS.

677. Electrolytes.—Liquids may be divided into three classes, depending upon their behavior towards the electrical current—those which do not conduct at all, as kerosene, turpentine, and oils generally; those which conduct without decomposition, *e.g.*, mercury and molten metals; those which are decomposed when they conduct a current, *e.g.*, solutions of acids or metallic salts and certain fused solid compounds. The liquids of the last class are called *electrolytes*, and the process of decomposing an electrolyte by

means of an electrical current is termed *electrolysis*. The two parts into which the electrolyte is decomposed are termed *ions*.

678. Electrolysis of Sulphuric Acid.—If a current of electricity flows into a solution of sulphuric acid (H_2SO_4) in water by means of an electrode, and if, after traversing the solution, it flows out through another electrode, then it will, by its passage, decompose the acid into two parts—H_2 and SO_4. *The hydrogen will appear*, in the form of gas bubbles, *at the electrode through which the current makes its exit from the solution.* The SO_4 will endeavor to appear at the electrode where the current entered, but the water of the solution seizes upon it, and together they form sulphuric acid, leaving, however, one portion of oxygen to appear, as gas, at the electrode. The effect of the passage of the current is to virtually decompose water (H_2O) into hydrogen and oxygen, there being twice as much of the former as of the latter.

FIG. 392.

Hoffmann's apparatus for electrolyzing sulphuric acid is shown in Fig. 392. The dilute acid solution is poured into the funnel *F*, and flowing into the two arms of the front U-tube fills them, providing the stop-cocks at their tops be opened. After filling, the cocks are closed and a current is made to pass between the two platinum electrodes *E E*. The gases which are evolved at the electrodes rise in the respective tubes above them and displace the liquid. These gases are subjected to the same pressure exerted by the liquid in the funnel. Their volumes may be read off from graduations on the tubes containing them. The gases may be taken off through the cocks and their natures tested—the oxygen being made to relight a glowing taper and the hydrogen being made to explode when mixed with air in a test-tube.

679. Metallic Salts.—When the electrolyte is a metallic salt solution *the metal will be deposited at the electrode where the current leaves the solution.* The acid of the salt appears at the other electrode. The metal may be deposited upon the surface of the electrode in the form of a thin metallic film. In case the metal has a strong affinity for the water of solution, *e.g.*, sodium in water, it will go into solution and hydrogen will be evolved as a secondary product. It is nevertheless true that Davy obtained metallic sodium and potassium by the electrolysis of strong caustic soda

and potash. These metals may be obtained by electrolysis, if a mercury electrode be employed. They then appear in the form of amalgams.

The character of a deposited metal often varies under different current strengths or different concentrations of solution. Copper may be deposited in the form of a black powder instead of an even metallic film. Silver may appear in the form of crystals. Platinum generally appears as a black, finely divided sponge. Tin, from tin chloride, forms a beautiful "tree" of tin crystals, the branches spreading out gracefully from the electrode.

680. Faraday's Laws.—Faraday proved that *a given quantity of electricity always deposits the same weight of a given ion* from an electrolyte through which it passes. Thus a coulomb of electricity always deposits .001118 gram of silver on an electrode. It makes no difference whether the electrolyte be molten silver iodide or chloride, or whether it be a water solution of silver nitrate, sulphate, acetate, or cyanide. The passage of one coulomb is always accompanied by the deposition of this much silver. *The weights of other chemical elements which a coulomb will deposit are in proportion to their chemical equivalents* This being so, it must be concluded that a given quantity of electricity ruptures the same number of molecular valencies, whatever the electrolyte may be.

681. Voltameters.—From Faraday's laws it will be readily seen that from weighing the amount of an ion, which is deposited by the passage of a certain quantity of electricity, this quantity may be determined. Thus, if a certain quantity deposits silver on an electrode so as to cause it to weigh 1.118 gram more than before the passage, it is evident that 1,000 coulombs have passed. If the quantity passed in the form of a *constant* current, which lasted for 1 second, then the current strength was 1,000 amperes. For an ampere means a strength of current which delivers 1 coulomb per second, but in this case 1,000 coulombs were delivered in a second. In general, if $z =$ the electro-chemical equivalent of the substance deposited, *i.e.*, grams per coulomb, $c =$ the current in amperes, $t =$ time in seconds that the current was maintained, the weight of the substance deposited

$$w = c\,z\,t.$$

In case w and t are measured, the current strength may be determined by the formula

$$c = \frac{w}{z\,t}.$$

Instruments for measuring current strengths in this manner are called *voltameters*. The substances generally employed for depo-

sition are copper from a solution of its sulphate, silver from its nitrate, and hydrogen from dilute H_2SO_4. In the case of hydrogen weighing is difficult, hence the volume is measured and then reduced to 760 mm. pressure and 0° C.

A current of 1 ampere deposits in 1 minute, of

Hydrogen (at 760 mm. and 0° C.)	6.942 cu. cm.
Copper	.01969 gram.
Silver	.06708 gram.
Zinc	.02018 gram.

The Edison electrical companies place zinc voltameters in the houses of their customers, and thus measure the quantity of electricity consumed.

682. Theory of Electrolysis.—The most satisfactory explanation of the phenomena of electrolysis is embodied in the theory of Grotthuss, somewhat modified by Clausius. The molecules of an ordinary solution are supposed to be in constant vibration in all possible directions. Owing to collisions between the molecules, or other causes, the constituent atoms are constantly leaving their partners and combining with others to form new molecules. Every molecule, having unit valency, is charged with the same quantity of electricity—half being positive and half negative. The positive resides on one ion of the molecule and the negative on the other. Now, upon subjecting the solution to a difference of potential between the electrodes, the *direction* of the molecular motions is controlled, and ions, which by chance are isolated, will tend to move towards one or the other electrode, according to the signs of the charges which are upon them. If the impressed electromotive force is large enough to prevent recombination of these ions, they will continue their movements towards the electrodes, and will accumulate around them. Upon touching the electrodes they impart to them their minute charges and the continuous accumulation of these maintains a current in the circuit.

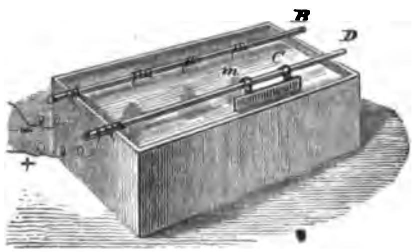
According to this theory, electrolytic conduction of electricity is similar to the convection of heat in liquids. The transportation of electricity is accompanied by a transportation of matter.

The remarkable connection between the results of the quantitative work of Faraday and the chemical equivalents of the elements, points to electrolysis as a fertile field for the investigation of the yet unknown nature of chemical affinity.

683. Electroplating.—The principles of electrolysis are made use of in the mechanic arts. Articles made of baser metals are covered over with a thin deposit of silver or gold and are said to

have been *electroplated.* The articles to be plated are suspended in a *bath* from a metallic rod, which is in electrical communication with the negative pole of a battery or dynamo (Fig. 393). The bath consists of a solution of some salt of the metal which is to be deposited, *e.g.*, silver or gold cyanide. The current from the positive pole of the dynamo enters the solution by means of an electrode, *C*, made of the same metal as that which is contained in the salt. Upon passing a current, the salt of the solution is decomposed—the metal depositing on the article to be plated, and the acid combining with the electrode *C* to form new salt, thus maintaining the concentration of the solution. The articles to be plated must be thoroughly scoured and cleansed before immersion in the bath. The character of the results obtained depends much upon the character and concentration of the baths and upon the magnitude of the currents and electro-motive forces employed. Full details must be looked for in technical books.

FIG. 393.



684. Electrotyping.—If the object to be plated consists of an impression, in wax or paper pulp, of the *type* from which a page is printed, the impression having been coated with fine plumbago to render it a good conductor, copper deposited upon it may be removed, and having been stiffened by melted lead (or some alloy) poured over its under surface, it may be used in the printing-press instead of the type. It is then called an *electrotype plate*, and when not in use may be preserved indefinitely for succeeding editions, while the type of which it is a copy can be distributed and used for other purposes.

685. Counter-Electromotive Force.—If a current be sent through a solution of alkaline zincate by means of two copper electrodes, zinc will be deposited on one electrode and the other will become oxidized. If the connections with the source of electricity be now removed and transferred to an electric bell, the bell will ring. The bath and electrodes have been transformed into a galvanic cell. The current which it gives is in a direction opposite to that which caused the decomposition of the solution. Its E. M. F. is about 0.79 volt, and is opposed to the original E. M. F. Had the original E. M. F. been less than this amount, no plating of the

electrodes could have occurred. The E. M. F. developed in the solution is termed a *counter-electromotive force.* It occurs in nearly all electrolytic actions, except when the electrodes are of the same metal as that which is being deposited, *e.g.,* copper in copper sulphate.

The counter-electromotive force developed in the electrolysis of dilute sulphuric acid is about 1.47 volt. Hence, to perform the electrolysis, more than one Daniell's cell is necessary.

The counter-electromotive force constitutes the polarization of a primary battery mentioned in Art. 634.

686. Storage Batteries.—The copper electrodes in alkaline zincate of the preceding article represent a very simple form of *storage battery* or *electrical accumulator.** Upon sending a current through it, the zinc is deposited and the battery is said to be *charged.* Some of the electrical energy has been transformed into chemical energy. The electrodes may be removed from the solution, packed away, and then be brought forward in the future and be made to turn back their energy into electricity. The electricity proper has not been stored away, but the energy represented by it.

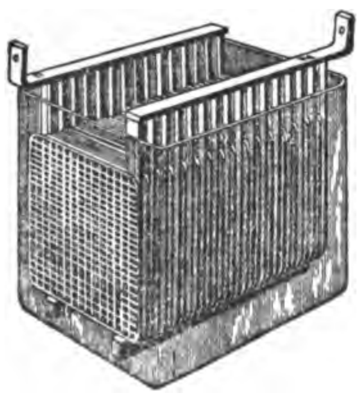
The first successful storage battery was constructed by Gaston Planté in 1860. His electrodes were made of sheet-lead, and the electrolyte was dilute sulphuric acid. In order to expose a large surface of electrodes he made them of large sheets which he coiled up into spirals, as shown in Fig. 394, the two plates being insulated from each other by rubber bands between the spirals. The object of the large surface was to increase the capacity of the cell. The spiral form was conducive to a small internal resistance. Upon the passage of a current the acid was decomposed and hydrogen reduced one electrode to bright metallic lead, while oxygen coated the other with peroxide of lead. These two conditions of the electrodes rendered them capable of giving an electro-motive force of two volts. By repeated charging in alternate directions the surfaces of both electrodes were rendered spongy, thus exposing an increased surface to the action of the ions and increasing the capacity of the cell accordingly. This preliminary alternate charging was termed by him "formation" of the electrodes, and was performed at the expense of costly currents.

FIG. 394.

In order to reduce the time and expense of formation, Faure used lead plates as a support and covered them with a paste made of powdered oxide of lead mixed with sulphuric acid. This paste he kept in place by covering the sheets with felt. When the

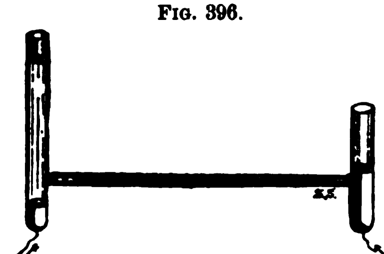
charging current was connected the oxide on one plate was changed to a higher oxide, and on the other plate transformed into metallic sponge. This idea of Faure was an excellent one, and is at the foundation of the construction of all the commercial lead accumulators. The percentage of energy recovered by discharge was greatly increased. His method of keeping the paste in place by felts was, however, soon abandoned, because fine lead needles soon filled up the interstices of the felt, and thus made a metallic connection between the electrodes. Holes were then punched in the lead plates and the paste pressed into them. A large number of the patents recently issued for accumulators refer to methods of making these holes and pressing in the paste, or to the shape of the holes themselves after they have been punched. The shapes vary from a slight depression on the surface to a hole completely through the plate, and even further, to a hollow plate, with small openings leading to the surface. A great deal depends upon this shape, for the paste changes its volume during the process of charging and discharging, and it would tend to loosen itself from some shaped openings and fall to the bottom of the cell, while in others it would tend to tighten itself, and thus provide a better contact.

FIG. 395.



A modern commercial storage battery is shown in Fig. 395. The electrodes are made up of a number of pasted plates, or *grids* as they are called. The grids of one electrode are alternated with those of the other, and are all connected by *lugs* with common cross-bars which constitute the terminals of cell.

FIG. 396.



687. Capillary Electrometer.—This instrument is for the measurement of small differences of potential not exceeding 1 volt. A simple form is represented in Fig. 396. It consists of two upright test-tubes connected by a horizontal capillary glass tube of about $\frac{1}{2}$ mm. internal diameter.

Into one of the test-tubes is poured mercury and into the other dilute sulphuric acid. The heights of the two liquids are so arranged that the dividing surface between them shall be in the horizontal tube. Upon subjecting the two liquids to an electro-motive force, applied at two platinum terminals fused into the bottoms of the test-tubes, an electrolytic action will be started at the point of the capillary tube where the acid meets the mercury. The surface tension will be accordingly modified and the balance between the two columns will be destroyed. To reproduce a balance the dividing surface must move along the capillary tube in one direction or the other, depending upon which liquid has the higher potential. The distance moved depends upon the potential difference and becomes a measure of it.

688. Light and Electricity.—At the present time many investigators are experimenting upon the close relation between the phenomena of light and those of electricity. Trustworthy results point to the fact that electricity is the luminiferous ether itself, as was previously stated. A motion of the ether is unrestrained in a perfect electrical conductor. In a dielectric only a limited displacement of the ether particles is possible, except in case the dielectric is ruptured. A displacement always subjects the enclosing dielectric to a strain, and can be produced by a neighboring conductor having an electrostatic charge or by its conveying an electrical current. The displacement resulting from a current is in a direction opposite to the current, and occurs through all the dielectric which surrounds the current. Upon starting the current the displacements near the conductor occur before those at a distance. The velocity of propagation of the first impulse causing displacement is the same as the velocity of light.

A full exposition of the ether hypothesis, and to what extent it explains electrical phenomena is, of course, out of place here.

689. Double Refraction from Electrostatic Strain.—Kerr showed that the strain in a dielectric, caused by electrostatic difference of potential, could be detected by means of polarized light. He placed a block of glass between two Nicol's prisms which served as analyzer and polarizer. Into opposite sides of the glass were bored two holes, not quite meeting each other, but separated by about 2 mm. Into these holes were placed wires, which were connected with the poles of a Holtz machine. Upon creating a difference of potential between the ends of the wires the glass was subjected to strain and exhibited to an eye placed at the analyzing Nicol similar colors to those given by mechanically strained glass. The glass was made doubly refracting.

690. Magneto-Optic Twisting of the Plane of Polarized Light.—Faraday discovered that the plane of polarization of a ray of light which traversed a magnetic field in a direction parallel to the lines of force was twisted by the field. One form of Faraday's experiment is to place a straight electro-magnet between two Nicol's prisms, which have been crossed so as to produce extinction of light. Substitute for the iron core of the magnet a tube with glass ends, which is filled with bisulphide of carbon. Before the magnet is excited a ray of light from the polarizer passes through the liquid and is brought to extinction by the analyzer. If, now, the magnet be excited by an electrical current, the analyzer no longer extinguishes the ray, and that it may do so must be rotated through a certain angle. The plane of the ray has been twisted or rotated by the magnetic field. The direction of the rotation is the same as the direction of the exciting current. By reversing the current the plane will be twisted in an opposite direction. The amount of the rotation of the analyzer necessary to reproduce extinction of the ray is directly proportional to the length of the tube and to the strength of the magnetic field, *i.e.*, to the strength of the exciting current. It also depends upon the nature of the liquid in the tube. In general it may be said that substances of high refractive indices have large rotatory powers.

As might be expected, rays of the different colors are rotated through different angles. Hence, if complete extinction by large rotations be desired, monochromatic light should be used.

691. Rotation of the Plane by Reflection.—Kerr discovered that the plane of polarization was rotated when the ray was reflected from the polished pole of the iron core of an electro-magnet. In this case the direction of rotation was contrary to the direction of the magnetizing currents.

692. Photo-Electric Properties of Selenium.—Selenium, when thoroughly annealed, offers a resistance to an electric current which is dependent upon the degree to which it is illuminated. An increase of illumination decreases the resistance. A piece of selenium, whose resistance in the dark was 500 ohms, has been known to decrease its resistance to 50 ohms upon exposure to bright sunlight.

This peculiarity of selenium is made use of by Bell in the construction of his *photophone*. This instrument is intended for transmitting sounds to a distance by means of rays of light, which are reflected from a mirror that is made to vibrate by the sounds. Light of varying intensity is made thus to impinge upon a piece of selenium, which is connected in circuit with a battery and a Bell

telephone receiver. The variations in the resistance of the selenium, because of the varied illumination, cause variations of the current in the receiver, which serve to reproduce the sounds.

Quite recently Shelford-Bidwell has exhibited an apparatus in which selenium is made to light the gas as darkness comes on and to turn it off as daylight appears.

Problems.

1. How much copper will be deposited by a current of 3 amperes in an hour?
2. A current of 0.5 ampere is used for preparing pure silver by electrolysis: how long must the current be allowed to flow in order to obtain a deposit of 4 grams?
3. What is the strength of a current which deposits a milligram of copper per minute?
4. It is found that a current of 1.868 ampere deposits 1.108 gram of copper in half an hour: what value does this give for the electro-chemical equivalent of copper?
5. What is the strength of a current which deposits 0.935 gram of copper in 1 hour and 10 minutes.

CHAPTER X.

THE RELATIONS BETWEEN ELECTRICITY AND HEAT.

693. Power of the Electrical Current.—A current whose strength is c carries in t seconds $c\,t$ units of electricity from a potential V to one of V'. The work which has to be expended in doing this is $c\,t\,(V' - V)$, as was shown in Art. 567. In this case $V' - V$ is equal to the electro-motive force E, which is sending the current. Hence, representing the work by A, we have

$$A = c\,t\,E.$$

If c, t, and E are measured in absolute units, the work is given in ergs.

The *power* of the current P being the rate at which the work is done, *i.e.*, the work divided by the time required to perform it is expressed by the formula

$$P = \left(\frac{A}{t}\right) = c\,E.$$

Expressing c and E in *amperes* and *volts* respectively will di-

vide the ergs per second by 10^7. This gives the power in *watts* (Art. 38).

Inasmuch as $c = \frac{E}{R}$ and $E = c\ R$, by Ohm's law, these values may be substituted, and we have, further,

$$P = \frac{E^2}{R} \text{ and}$$
$$P = c^2\ R.$$

694. Heat Developed in a Conductor.—Whenever the energy which is represented by a current is not expended in doing external work, as in driving motors or decomposing electrolytes, it is transformed into heat. The conductor which carries the current becomes heated. If a conductor of resistance, R, carries a current c, then, by Ohm's law, the difference of potential between its ends, $E = c\ R$. The energy represented by the current is, as in the preceding article,

$$A = c\ t\ E = c^2\ R\ t \text{ ergs.}$$

This energy is transformed into heat. To express the heat in gram-calories, Joule's mechanical equivalent of heat must be introduced. Without going through with the transformations it is sufficient to say that a current of c amperes flowing for t seconds through R ohms communicates to the conductor carrying it

$$H = c^2\ R\ t\ 0.24 \text{ gram-calories.}$$

695. Rise in Temperature of the Conductor.—A long thick wire could have the same resistance as a short thin one, but a given current traversing them for a given time would produce the same quantities of heat in each. The short thin wire, not weighing so much, might have its temperature raised several hundred degrees, while the thick wire would suffer a rise of a few degrees only.

In order to determine what rise in temperature will accompany a given quantity of heat imparted, account must be taken of the dimensions of the conductor, the specific heat of the substance of which the conductor is composed, and the temperature coefficient of the conductor, *i.e.*, the amount by which its resistance would increase under a rise of one degree of temperature. A full consideration cannot be considered in these chapters. It is well to know, however, that, *in different wires of the same material, traversed by the same current, the rise in temperature is inversely proportional to the fourth power of their diameters.*

A wire of given resistance, traversed by a given constant current, will receive the same amount of heat each second that the current flows. After a short time the temperature of the wire may rise to

such a point that it gives off to surrounding objects, by radiation and conduction, just as much heat as it receives in every second. The temperature then remains constant at this point as long as the flow is maintained.

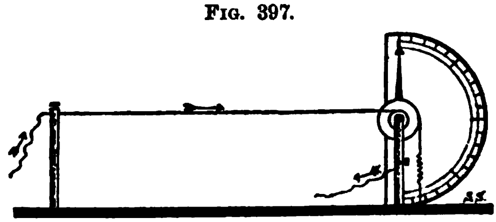
The heat effects mentioned may be illustrated by sending a strong current through a chain, whose alternate links are made of platinum and silver wire. The platinum links will be heated to luminosity while the appearance of the silver remains unaltered. The reason for this is that the platinum offers a much greater resistance than the silver, and its specific heat is less.

Platinum wires, heated red-hot by currents, are much used by surgeons for cauterization. They are much easier of manipulation than the knife.

696. Hot Wire Ammeters and Voltmeters.—The expansion in length which a wire undergoes when its temperature is raised to a certain point by a current which traverses it, can be made a measure of the strength of the current. A given wire has a definite length at a given temperature. Increasing the temperature increases the length. Every current produces a definite length in the wire. Different current strengths correspond to different lengths. A measurement of the length can thus be made a measure of the current strength.

A simple *ammeter*, whose action depends upon this principle, is represented in Fig. 397. The current to be measured is passed

FIG. 397.



through a long and thin platinum or iron wire, one of whose ends is clamped in a stationary binding-post. The other end passes around and is fastened to a small metallic cylinder. This cylinder turns upon a metallic pivot fastened in another binding-post. The current having traversed the wire leaves it by this binding-post. The wire is subjected to a constant strain, exerted by a spiral spring attached to the periphery of a disc, which is fastened to one end of the cylinder. The disc carries a radial pointer, whose end moves over a graduated scale whenever the length of the wire is changed by a change in temperature caused by a current. The

graduation of the scale is empirical, being determined by the assistance of some other current measurer.

As the current strength is dependent upon the difference of potential between the two binding-posts, it is evident that the instrument may be graduated as a *voltmeter*, *i.e.*, will indicate the volts impressed upon it. As it is not desirable that a large current should flow through a voltmeter, the wire of such an instrument should have a large resistance. The voltmeters of Cardew are constructed on this principle. Sometimes a high resistance coil is inserted in series with the wire, and then the voltmeter readings indicate the fall in potential between the terminals of the spool and wire in series.

Hot wire ammeters and voltmeters can be employed to measure currents and voltages which rapidly alternate their directions. For the heat produced being dependent on the square of the current strength is *positive*, whether the current flows in a positive or negative direction.

697. Electric Welding. — The welding together of two pieces of metal, by means of the electric current, as done in the Thomson process, depends upon the heat produced. The pieces are pressed together and a powerful current (sometimes 50,000 amperes) is sent across the juncture. The consequent heat renders the metal plastic, and upon cooling a most perfect joint is obtained.

FIG. 398.

698. The Electric Arc. — If two rods of carbon, traversed by a current from a source of at least 40 volts electromotive force, be touched together at their ends and then be separated by a few millimeters' distance, an electric flame or arc will be observed to pass over this distance. A brilliant light will accompany it, the extreme brilliancy being at the end surfaces of the rods. If allowed to burn for a few moments the rods and flame will present an appearance like that represented in Fig. 398. The end of the positive rod will have formed itself into a sort of crater, while the end of the negative will have become pointed. If allowed to burn for some time, the rods will be consumed, and, in a

given time, about twice as much of the positive rod will be consumed as of the negative.

In order to form an *arc* it is necessary that the points be at first in contact. When in loose contact the current encounters a great resistance, and accordingly heats the points until a temperature is reached which is sufficient to vaporize the carbon. Carbon vapor is a much better conductor of electricity than air, and whereas an arc could not be maintained across an air space, yet it can be across a space filled with this vapor.

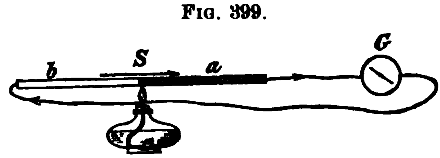
The heat at the vapor portion of the arc is intense, being sufficient to vaporize the most refractory substances, of which carbon itself is the best example. The heat at the crater, though not so intense, is the cause of greater illumination, because of being associated with a solid instead of a vapor.

Recent investigations, concerning the fall of potential along the arc, indicate that a large portion of the electrical energy represented by it is consumed in maintaining the heat of the crater.

699. Incandescent Electric Lamps.—These lamps consist of filaments of carbonized bamboo, paper, or silk, which are heated to incandescence by the current. That the filaments may not be consumed by combustion, they are sealed into glass bulbs, from which the air has been exhausted. Although no oxygen is present, the filaments become disintegrated by continuous use. Particles of carbon escape from the surface of the filament and are oftentimes deposited upon the interior of the bulb, causing a brownish opalescent appearance.

700. Thermo-Electricity.—Let two bars of bismuth (b) and antimony (a) be soldered together as in Fig. 399. If, now, the joint S be heated by a lamp a current will flow across the heated junction from the bismuth to the antimony, as will be shown by the galvanometer G.

FIG. 399.



The electro-motive force of the current depends upon the metals in contact at the heated junction. If any one of the metals given below be joined with any one following it in the list, upon applying heat the current will flow across the junction from the former to the latter: Bismuth, lead, platinum, tin, zinc, copper, iron, antimony.

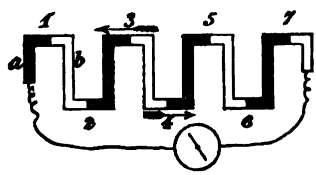
The thermo-electro-motive force is proportional to the difference of temperature between the junction and the rest of the circuit.

The E. M. F. of a single thermo-element is very small. If the junction of a copper-iron element be heated 1° C. above the temperature of the rest of the circuit, the E. M. F. developed is about fourteen millionths of a volt.

In some cases, *e.g.*, with iron, a continued increase of temperature at the junction finally reverses the direction of the current.

701. Thermo-Electric Pile.—If a series of bars of bismuth and antimony be arranged, as in Fig. 400, and the junctions marked 3 and 4 be equally heated, no current will be indicated by the galvanometer; for the flow at 3 would be from the bismuth to the antimony as indicated by the arrow, while at 4 it would also be from *b* to *a*, as shown, and these two currents would neutralize each other.

FIG. 400.



But if we heat only one set of junctions, the odd-numbered for instance, then a current flows whose electro-motive force is proportional to the number of heated junctions.

A set of twenty or thirty pairs, conveniently arranged so that the alternate junctions may be simultaneously subjected to heating or cooling effects, is called a *thermo-pile*, and has been an important instrument in investigations upon radiant heat.

702. Peltier Effect.—Peltier discovered a phenomenon which is the converse of that mentioned in the preceding articles. He found that, if a current of electricity be sent through a junction of dissimilar metals, the junction becomes heated or *cooled* according to the direction of the current. For instance, if a current be sent through a junction from bismuth to antimony, the junction will absorb heat, *i.e.*, become cooled. If the current be reversed the junction will become heated.

The heat thus produced is not owing to the resistance of the conductors. For the heat from resistance is not altered by a change in the direction of the flow of the current. Cooling can never result from ohmic resistance. Again, the heat of the Peltier effect is proportional to the current strength simply, whereas the heat from resistance is proportional to the square of the current strength.

Problems.

1. An 11,000 watt dynamo develops an E. M. F. of 110 volts: (*a*) What is the current strength in the mains? (*b*) How many incandescent lamps, of 220 ohms hot resistance, will it light, provid-

ing they are arranged in multiple arc? (c) How many gram-calories will be developed in each lamp per second? (d) How many watts will be consumed by each lamp?

Ans. (a) 100 amperes. (b) 200 lamps. (c) 13.2 calories. (d) 55 watts.

2. How much power is required to properly operate an arc lamp which carries 10 amperes and has a difference of potential of 45.2 volts between its terminals?

3. How many calories are developed per minute in a wire of 100 ohms resistance, traversed by 5 amperes?

4. A wire of 2 ohms resistance placed in 100 grams of water is traversed by a certain current, which, in 20 minutes, raises the temperature of the water from 18° to 28° C.: what is the current strength? *Ans.* 1.32 amperes, nearly.

The Electrical Units.

Electrical magnitudes may be expressed in three different sets of units. Two of them—the *absolute electrostatic* and the *absolute electro-magnetic* units—are termed absolute because they are units derived from the absolute units (Art. 4) of length, mass, and time, viz., the centimetre, gram, and second. The third set are called *practical* units, because they are the ones which are employed by practical electricians. They are either decimal multiples or decimal parts of the electro-magnetic units.

Electrostatic Units.

The Unit of Quantity of electricity is that quantity which, when placed at a distance of one centimetre from a similar and equal quantity, repels it with a force of one dyne (Art. 563).

The Unit Strength of Current flows in a circuit when a unit quantity of electricity passes any section of the conductor in one second.

The Unit Difference of Potential exists between two points when it requires an expenditure of one erg of work to bring a unit quantity of electricity from one point to the other against the electric force.

The Unit of Resistance is offered by that conductor which, when interposed between two bodies whose potentials are maintained at a constant difference of unity, allows a unit current to pass along it.

The Unit of Capacity is possessed by that conductor which requires that it be charged with a unit quantity of electricity in order that its potential may be raised from zero to unity.

Electro-magnetic Units.

The Unit Strength of Current is such that, when flowing through

a conductor of one centimetre length which is bent into an arc of one centimetre radius, it will exert a force of one dyne on a unit magnetic pole situated at the centre.

The Unit Quantity of electricity passes in one second through a section of a conductor which is traversed by a current of unit strength.

The Unit Difference of Potential (*or of Electro-motive Force*) exists between two points when it requires the expenditure of one erg of work to bring a unit of electricity from one point to the other against the electric force.

The Unit of Resistance is offered by that conductor which, when interposed between two bodies whose potentials are maintained at a constant difference of unity, allows a unit current to pass along it.

The Unit of Capacity is possessed by that conductor which requires that it be charged with a unit quantity of electricity in order that its potential may be raised from zero to unity.

A little consideration will show that in the electrostatic and electro-magnetic systems the definitions of all the units except that for quantity are identical. Whereas the electrostatic unit of quantity is determined from its exerting a dyne of force on another unit quantity, the electro-magnetic unit of quantity is determined from its exerting a dyne of force, when moving as a current, on a unit magnetic pole. The electro-magnetic unit is about 3×10^{10} times the electrostatic unit. This numerical factor is the same as the velocity of the propagation of light expressed in centimetres per second. This fact, combined with certain mathematical relations which exist between the two units, is of great significance in sustaining the ether theory of electricity.

PRACTICAL UNITS.

Many of the absolute units would be inconveniently large and others would be inconveniently small for practical use. Therefore the following units, based upon the electro-magnetic units, are used:

Electromotive force	*Volt*	$= 10^8$	electro-magnetic	units.
Resistance	*Ohm*	$= 10^9$	"	"
Current................	*Ampere*	$= 10^{-1}$	"	"
Quantity...............	*Coulomb*	$= 10^{-1}$	"	"
Capacity...............	*Farad*	$= 10^{-9}$	"	"

Even these units are not of a magnitude suited for the use of all electricians. Thus a physician uses currents whose strengths can be more easily expressed in thousandths of an ampere. The prefix *milli-* is therefore used for "one thousandth" and a *milliampere* is the thousandth part of one ampere. Capacities are best expressed in millionths of a farad or *microfarads*. The high resistances offered by insulations are conveniently expressed in *megohms* = one million ohms.

APPENDIX.

APPLICATIONS OF THE CALCULUS.

I. Fall of Bodies.

1. Differential Equations for Force and Motion.—These are three in number, as follows:

$$1. \quad v = \frac{d\,s}{d\,t}.$$

$$2. \quad f = \frac{d\,v}{d\,t} = \frac{d^2\,s}{d\,t^2}.$$

$$3. \; f\,d\,s = v\,d\,v.$$

These equations are readily derived from the elementary principles of mechanics. In Art. 6 we have $v = \frac{s}{t}$. Reducing the numerator and denominator to infinitesimals, v remains finite, and the equation becomes $v = \frac{d\,s}{d\,t}$; which is Equation 1st. Therefore, if the space described by a body is regarded as a function of the time, the *first* differential coefficient expresses the *velocity.*

Again (Art. 12), $f = \frac{v}{t}$, where f represents a *constant* force. Making velocity and time infinitely small, we get the intensity of the momentary force, $f = \frac{d\,v}{d\,t}$. But, by Equation 1st, $v = \frac{d\,s}{d\,t}$; $\therefore f = \frac{d^2\,s}{d\,t^2}$; which is Equation 2d. Hence we learn that the *first* differential coefficient of the *velocity* as a function of the time, or the *second* differential coefficient of the *space* as a function of the time, expresses the *force.*

Equation 3d is obtained by multiplying the 1st and 2d crosswise, and removing the common denominator.

We proceed to apply these equations to the preparation of formulæ for falling bodies.

2. Bodies falling through Small Distances near the Earth's Surface.—In this case, let the accelerating force, which

is considered *constant*, be called g. Then, by Eq. 2, $g = \frac{dv}{dt}$, $\therefore dv = g\,dt$. Integrating, we have $v = gt + C$. But, since $v = 0$ when $t = 0$, $\therefore v = gt$, and $t = \frac{v}{g}$, as in Art. 27.

Again, substituting gt for v in Eq. 1, $ds = gt\,dt$; and by integration, $s = \frac{1}{2}gt^2 + C$; but $C = 0$, for the same reason as before; $\therefore s = \frac{1}{2}gt^2$, and $t = \sqrt{\frac{2s}{g}}$.

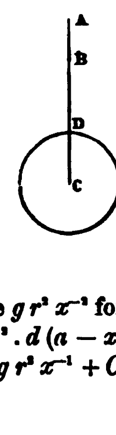
Once more, equating the two foregoing values of t, we have $v = \sqrt{2gs}$, and $s = \frac{v^2}{2g}$.

If, in the equation, $s = \frac{1}{2}gt^2$, v be substituted for gt, we have $s = \frac{1}{2}vt$, or $vt = 2s$; that is, the acquired velocity multiplied by the time of fall gives a space twice as great as that fallen through (Art. 21).

3. Bodies falling through Great Distances, so that Gravity is Variable, according to the Law in Art. 16.—

Suppose a body to fall from A to B (Fig. 1), toward the centre C. Let $AC = a$; $BC = x$; $DC = r$, the radius of the earth.

FIG. 1.



The force f at B, is found by the principle, Art. 16,

$$x^2 : r^2 :: g : f = gr^2\frac{1}{x^2} = gr^2x^{-2}.$$

4. To find the Acquired Velocity.—Substitute gr^2x^{-2} for f, and $a - x$ for s, in Equation 3d, and we have $gr^2x^{-2}\,.\,d(a-x) = v\,dv$; $\therefore$ by integration $\frac{1}{2}v^2 = \int -gr^2x^{-2}\,dx = gr^2x^{-1} + C$. But $v = 0$, when $x = a$; $\therefore C = -gr^2a^{-1}$; and

$$\tfrac{1}{2}v^2 = gr^2x^{-1} - gr^2a^{-1};$$

$$\therefore \quad v^2 = \frac{2gr^2(a-x)}{ax};$$

$$\therefore \quad v = \left\{\frac{2gr^2(a-x)}{ax}\right\}^{\frac{1}{2}}$$

This is the general formula for the acquired velocity. If the body falls to the earth, $x = r$, and the formula becomes

$$v = \left\{\frac{2gr(a-r)}{a}\right\}^{\frac{1}{2}}$$

Again, if the body falls to the earth through so small a space that $\frac{r}{a}$ may be regarded as a unit, the formula reduces to

$$v = \{2g(a-r)\}^{\frac{1}{2}} = (2gs)^{\frac{1}{2}};$$

the same as obtained by other methods.

If a body falls to the earth from an infinite distance, it does not acquire an infinite velocity. For then, as we may put a for $a-r$,

$$v = \left\{\frac{2gr \cdot a}{a}\right\}^{\frac{1}{2}} = (2gr)^{\frac{1}{2}} =$$

$$(2 \cdot 32\tfrac{1}{6} \cdot 3956 \cdot 5280)^{\frac{1}{2}} \text{ feet} = 6.95 \text{ miles.}$$

Therefore, the greatest possible velocity acquired in falling to the earth is less than *seven miles;* and a body projected upward with that velocity would never return.

5. To find the Time of Falling.—From equation first we obtain $dt = \frac{ds}{v}$; in this, substitute $d(a-x)$ for ds, and $\frac{\{2gr^2(a-x)\}^{\frac{1}{2}}}{(ax)^{\frac{1}{2}}}$ for v, as found in the preceding article; then

$$dt = \frac{(ax)^{\frac{1}{2}} \cdot d(a-x)}{\{2gr^2(a-x)\}^{\frac{1}{2}}} = \left(\frac{a}{2gr^2}\right)^{\frac{1}{2}} \cdot \frac{-x^{\frac{1}{2}}dx}{(a-x)^{\frac{1}{2}}};$$

$\therefore$ by integration $t = \left(\frac{a}{2gr^2}\right)^{\frac{1}{2}} \cdot \int -x^{\frac{1}{2}}dx(a-x)^{-\frac{1}{2}}$.

By the formula in the calculus for reducing the index of x we obtain

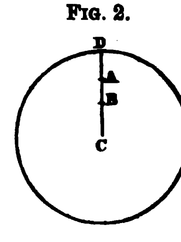
$$\int -x^{\frac{1}{2}}dx(a-x)^{-\frac{1}{2}} = (ax-x^2)^{\frac{1}{2}} - \frac{a}{2}\text{vers}^{-1}\left(\frac{2x}{a}\right) + C.$$

Now, when $t = 0, x = a$; $\therefore C = \frac{a\pi}{2}$;

hence, $t = \left(\frac{a}{2gr^2}\right)^{\frac{1}{2}}\left\{(ax-x^2)^{\frac{1}{2}} - \frac{a}{2}\text{vers}^{-1}\left(\frac{2x}{a}\right) + \frac{a\pi}{2}\right\}$.

6. Bodies falling within the Earth (supposed to be of uniform density), where Gravity Varies as the Distance from the Centre.—

FIG. 2.



Suppose a body to fall from A to B (Fig. 2); and let $DC = r$, $AC = a$, and $BC = x$. Then

$$r : x :: g : f = \frac{g}{r}x = \text{force at B.}$$

To find the velocity acquired.—By Eq. 3d,

$$v\,d\,v = f\,d\,s;\ \therefore v\,d\,v = \frac{g}{r}x\,.\,d\,(a - x) = -\frac{g\,x\,d\,x}{r};$$

$$\therefore \tfrac{1}{2}v^2 = -\frac{g\,x^2}{2\,r} + C;\ \text{but } v = 0 \text{ when } x = a;$$

$$\therefore C = \frac{g\,a^2}{2\,r}, \text{ and } \tfrac{1}{2}v^2 = \frac{g\,(a^2 - x^2)}{2\,r};\ \therefore v = \left\{\frac{g}{r}(a^2 - x^2)\right\}^{\frac{1}{2}}.$$

If the body falls from the surface to the centre, $x = 0$, and this formula becomes $v = (g\,r)^{\frac{1}{2}} = (32\tfrac{1}{6} \times 3956 \times 5280)^{\frac{1}{2}} = 25{,}904$ feet per second.

To find the time of falling.—By Equation 1st, and substitutions, we obtain $d\,t = \frac{d\,s}{v} = \frac{d\,(a-x)}{v} = -\frac{d\,x}{v} = \frac{-\,d\,x}{\left\{\frac{g}{r}(a^2 - x^2)\right\}^{\frac{1}{2}}}$

$$= \left(\frac{r}{g}\right)^{\frac{1}{2}} \times \frac{-\,d\,x}{(a^2 - x^2)^{\frac{1}{2}}};\ \therefore t = \left(\frac{r}{g}\right)^{\frac{1}{2}}\int\frac{-\,d\,x}{(a^2-x^2)^{\frac{1}{2}}} = \left(\frac{r}{g}\right)^{\frac{1}{2}}\cos^{-1}\frac{x}{a} + C$$

When $t = 0, x = a, \frac{x}{a} = 1$, and the arc, whose cosine is $1 = 0$;

$$\therefore C = 0.\quad \therefore t = \left(\frac{r}{g}\right)^{\frac{1}{2}} \times \cos^{-1}\frac{x}{a}.$$

If the body falls to the centre, $x = 0$, and $t = \left(\frac{r}{g}\right)^{\frac{1}{2}} \times \frac{\pi}{2}$; in which a does not appear at all; so that the time of falling to the centre from any point within the surface is the same; and equals $\left(\frac{3956 \times 5280}{32\frac{1}{6}}\right)^{\frac{1}{2}} \times 1.570796$ in seconds, or 21m. 5.8s.

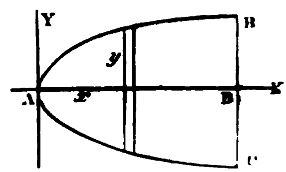
II. Centre of Gravity.

7. Principle of Moments.—In order to apply the processes of the calculus to the determination of the centre of gravity, the principle is used, which was proved (Art. 78), that if every particle of a body be multiplied by its distance from a plane, and the sum of the products be divided by the sum of the particles, the quotient is the distance of the common centre from the same plane. The product of any particle or body by its distance from the plane, is called its *moment* with respect to that plane.

8. General Formulæ.—Let $B\,A\,C$ (Fig. 3) be any symmetrical curve, having $A\,X$ for its axis of abscissas, and $A\,Y$, at right

angles to it, for its axis of ordinates. It is obvious that the centre of gravity of the line $B\,A\,C$, of the area $B\,A\,C$, of the solid of revolution around the axis $A\,X$, and of the surface of the same solid, are all situated on $A\,X$, on account of the symmetry of the figure. It is proposed to find the formula for the distance of the centre from $A\,Y$, in each of these cases. Let G in every instance represent the distance of the general centre of gravity from the axis $A\,Y$, or the plane $A\,Y$, at right angles to $A\,X$. The distance G would plainly be the same for the *half* figure $B\,A\,D$, as for the whole $B\,A\,C$; expressions may therefore be obtained for either, according to convenience.

FIG. 2.



1. *The line* $A\,B$.—Let x be the abscissa, and y the ordinate; then $(d\,x^2 + d\,y^2)^{\frac{1}{2}}$ is the differential of the line $A\,B$. For brevity, let s = the line, and $d\,s$ its differential. If we now multiply this differential by its distance from $A\,Y$, $x\,d\,s$ is the moment of a minute portion of the line; and the integral of it, $\int x\,d\,s$, is the moment of the whole. Dividing this by the line itself, i. e. by s, we have $\frac{\int x\,d\,s}{s}$ for the distance G.

2. *The area* $B\,A\,D$.—The differential of the area is $y\,d\,x$; the differential of its moment is $x\,y\,d\,x$; hence the moment itself is $\int x\,y\,d\,x$; and the distance $G = \frac{\int x\,y\,d\,x}{\text{area}}$.

3. *The solid of revolution.*—The differential of the solid, generated by the revolution of $A\,B$ on $A\,X$, is $\pi\,y^2d\,x$; the differential of its moment is $\pi\,x\,y^2d\,x$; and the moment is $\int \pi\,x\,y^2d\,x$; hence the distance $G = \frac{\int \pi\,x\,y^2d\,x}{\text{solid}}$.

4. *The surface of revolution.*—The differential of the surface is $2\,\pi\,y\,d\,s$; the differential of its moment is $2\,\pi\,x\,y\,d\,s$; and therefore the moment is $\int 2\,\pi\,x\,y\,d\,s$; and the distance $G = \frac{\int 2\pi\,x\,y\,d\,s}{\text{surface}}$.

9. Application of Formulæ.—We proceed to determine the centre of gravity in a few cases by the aid of these formulæ:

1. *A straight line.*—Imagine the line placed on $A\,X$, with one extremity at the origin A. The moment of a minute part of it is $x\,d\,x$, and that of the whole is $\int x\,d\,x$, while the length of the whole is x; $\therefore\ G = \frac{\int x\,d\,x}{x} = \frac{\frac{1}{2}\,x^2 + C}{x} = \frac{1}{2}\,x$, as it evidently should

be. In all the cases considered here, $C = 0$, because the function vanishes when x does.

2. *The arc of a circle.*—By formula 1st we have $G = \frac{\int x\,d s}{s}$; but $d s = (d x^2 + d y^2)^{\frac{1}{2}}$ and the equation of the circle is $y^2 = a^2 - x^2$.

By differentiating, $d y^2 = \frac{x^2\,d x^2}{y^2} = \frac{x^2\,d x^2}{a^2 - x^2}$,

and $$d x^2 = \frac{(a^2 - x^2)\,d x^2}{a^2 - x^2};$$

$$\therefore\ d s = (d x^2 + d y^2)^{\frac{1}{2}} = \frac{a\,d x}{\sqrt{a^2 - x^2}};$$

$$\therefore\ G = \frac{\int x\,d s}{s} = \frac{a}{s}\int\frac{x\,d x}{\sqrt{a^2 - x^2}} = \frac{a}{s}\int d y = \frac{a\,y}{s}.$$

If the arc be doubled and called t, and its chord be represented by $c\ (= 2\,y)$, then the distance from the centre of the circle to the centre of gravity of the arc, is $\frac{a\,c}{t}$, which is a fourth proportional to the *arc*, the *chord*, and the *radius*.

When the arc is a semi-circumference, $c = 2\,a$, and $t = \pi\,a$; $\therefore$ the distance of the centre of gravity of a semi-circumference from the centre of the circle is $\frac{2\,a}{\pi}$.

3. *The area of a circular sector.*—Suppose the given sector to be divided into an infinite number of sectors; then each may be considered a triangle, and its centre of gravity therefore distant from the centre of the circle by the line $\frac{2\,a}{3}$. Hence the centres of gravity of all the sectors lie in a circular arc, whose radius is $\frac{2\,a}{3}$; so that the centre of gravity of the whole sector coincides with the centre of gravity of that arc. The distance of the centre of gravity of the arc from the centre of the circle, by the preceding case, is $\frac{2}{3}a \times \frac{2}{3}c \div \frac{2}{3}t = \frac{2\,a\,c}{3\,t}$, which is therefore the distance of the centre of gravity of the sector from the centre of the circle. When the sector is a semicircle the distance becomes $\frac{2\,a \times 2\,a}{3\,\pi\,a}$ $= \frac{4\,a}{3\,\pi}$

4. *The area of a parabola.*—The equation of the curve is

$$y^2 = p\,x, \text{ or } y = p^{\frac{1}{2}}\,x^{\frac{1}{2}};$$

therefore the formula 2 for moment,

$$\int x\,y\,d\,x = \int p^{\frac{1}{2}}\,x^{\frac{3}{2}}\,d\,x = \tfrac{2}{5}p^{\frac{1}{2}}\,x^{\frac{5}{2}}\,(+C = 0);$$

but the area of the half parabola $= \frac{2}{3}\,p^{\frac{1}{2}}\,x^{\frac{3}{2}}$;

$$\therefore\; G = \tfrac{2}{5}\,p^{\frac{1}{2}}\,x^{\frac{5}{2}} \div \tfrac{2}{3}\,p^{\frac{1}{2}}\,x^{\frac{3}{2}} = \tfrac{3}{5}\,x.$$

To find the distance of the centre of gravity of the semi-parabola from the axis $A\,X$, proceed as follows: The differential of the area, as before, equals $y\,d\,x$; and the distance of its centre from $A\,X$ is $\frac{1}{2}\,y$. Therefore its moment with respect to $A\,X$ is $\frac{1}{2}\,y^2\,d\,x = \frac{1}{2}\,p\,x\,d\,x$; and the moment of the whole is $\int\frac{1}{2}\,p\,x\,d\,x = \frac{1}{4}\,p\,x^2$; $\therefore$ the distance of the centre from

$$A\,X = \tfrac{1}{4}\,p\,x^2 \div \tfrac{2}{3}\,p^{\frac{1}{2}}\,x^{\frac{3}{2}} = \tfrac{3}{8}\,p^{\frac{1}{2}}\,x^{\frac{1}{2}} = \tfrac{3}{8}\,y.$$

5. *The area of a circular segment.*—The equation of the circle is, $y = (2\,a\,x - x^2)^{\frac{1}{2}}$. Therefore (formula 2),

$$\int x\,y\,d\,x = \int x\,(2\,a\,x - x^2)^{\frac{1}{2}}\,d\,x.$$

Add and subtract $a\,(2\,a\,x - x^2)^{\frac{1}{2}}\,d\,x$, and it becomes

$$\int a\,(2\,a\,x - x^2)^{\frac{1}{2}}\,d\,x - \int(a - x)\,(2\,a\,x - x^2)^{\frac{1}{2}}\,d\,x =$$

$$a\int y\,d\,x - \frac{(2\,a\,x - x^2)^{\frac{3}{2}}\,(a - x)\,d\,x}{\frac{3}{2}\,(2\,a - 2\,x)\,d\,x} = a\,.\,\text{area } A\,B\,D - \tfrac{1}{3}\,(2\,a\,x - x^2)^{\frac{3}{2}}.$$

$$\therefore\; G = a - \frac{(2\,a\,x - x^2)^{\frac{3}{2}}}{3 \text{ area } A\,B\,D}.$$

When $x = a$, $G = a - \dfrac{4\,a}{3\,\pi}$; and the distance of the centre of gravity of a semicircle from the centre of the circle $= \dfrac{4\,a}{3\,\pi}$. When $x = 2\,a$, $G = a$, as it plainly should be.

6. *A spherical segment.*—The equation of the circle is $y^2 = 2\,a\,x - x^2$. Therefore (formula 3),

$$\int\pi x y^2 d\,x = \int\pi\,x\,d\,x\,(2ax - x^2) = \int 2a\pi x^2 dx - \int\pi x^3 dx = \tfrac{2}{3}a\pi x^3 - \tfrac{1}{4}\pi x^4;$$

$$\therefore\; G = \frac{\frac{2}{3}\,a\,\pi\,x^3 - \frac{1}{4}\,\pi\,x^4}{a\,\pi\,x^2 - \frac{1}{3}\,\pi\,x^3} = \frac{8\,a\,x - 3\,x^2}{12\,a - 4\,x}.$$

When $x = a$, $G = \frac{5}{8}\,a$; that is, the centre of gravity of a hemisphere is $\frac{5}{8}$ of radius from the surface, or $\frac{3}{8}$ of radius from the centre of the sphere. If $x = 2\,a$, $G = a$.

7. *A right cone.*—In this case $A\,B$ (Fig. 3), is a straight line, and its equation is $y = a\,x$, where a is any constant.

$\therefore\ y^2 = a^2x^2;\ \therefore\ \int \pi x y^2 dx = \int \pi a^2 x^3 dx = \frac{\pi}{4}a^2x^4;\ \therefore\ G = \frac{\frac{1}{4}\pi a^2 x^4}{\frac{1}{3}\pi a^2 x^3} = \frac{3}{4}x.$

Hence the centre of gravity of a cone is three-fourths of the axis from the vertex. See Art. 75.

8. *The convex surface of a right cone.*—The equation is

$$y = a x;\ \therefore\ d y^2 = a^2 d x^2;\ \text{and}\ (d x^2 + d y^2)^{\frac{1}{2}} = (a^2 + 1)^{\frac{1}{2}} d x.$$

Therefore (formula 4),

$$\int 2\pi x y ds = \int 2\pi x y (dx^2 + dy^2)^{\frac{1}{2}} = \int 2\pi a x^2 (a^2+1)^{\frac{1}{2}} dx = \tfrac{2}{3}\pi a x^3 (a^2+1)^{\frac{1}{2}}$$

= the moment of the surface. The surface itself,

$$= \pi y (x^2 + y^2)^{\frac{1}{2}} = \pi a x^2 (a^2 + 1)^{\frac{1}{2}}.\quad \therefore\ G = \frac{\frac{2}{3}\pi a x^3 (a^2+1)^{\frac{1}{2}}}{\pi a x^2 (a^2+1)^{\frac{1}{2}}} = \tfrac{2}{3} x.$$

The centre of gravity of the convex surface of a right cone is on the axis, at a distance equal to two-thirds of its length from the vertex.

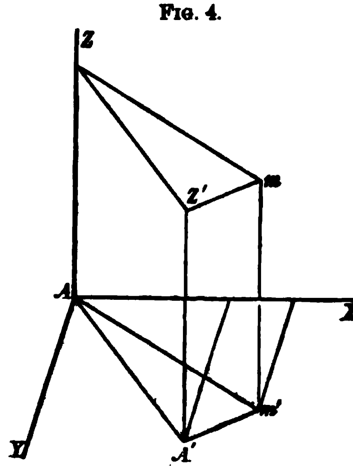
III. Centre of Oscillation.

10. To find the Moment of Inertia of a Body for any given Axis.—To render the formula $l = \frac{S(m r^2)}{M k}$ suitable to the application of the calculus, we have simply to substitute the sign of integration for S, and $d\,M$ for m, and we have

$$l = \frac{\int r^2\, d\,M}{M k}. \qquad (1)$$

It is useful to know how to find the ***moment of inertia*** with respect to any axis by means of the *known moment* with respect to *another* axis parallel to it and passing through the centre of gravity of the body.

FIG. 4.



Let $A\,Z$ (Fig. 4) be the axis passing through the centre of gravity of the body for which the moment of inertia is $\int r^2 dM$, and let $A'\,Z'$ be the axis parallel to it, for which the moment of inertia, $\int r'^2\, d\,M$ of the same mass M, is to be determined. For every particle m of the body the corresponding value of $A\,m'$ is $r^2 = x^2 + y^2$. In like man-

ner, if we denote the co-ordinates of A' by a and β, and the distance between the axes by a, we shall have $a^2 = a^2 + \beta^2$. Now the distance of the particle m from $A'\,Z'$ is $r'^2 = (x - a)^2 + (y - \beta)^2 = x^2 + y^2 + a^2 + \beta^2 - 2\,a\,x - 2\,\beta\,y = r^2 + a^2 - 2\,a\,x - 2\,\beta\,y$; $\therefore$ $\int r'^2\,d\,M = \int r^2\,d\,M + a^2 \int d\,M - 2\,a \int x\,d\,M - 2\,\beta \int y\,d\,M = a^2\,M + \int r^2\,d\,M$, ... (2)
since $A\,Z$ passes through the centre of gravity of the body. Hence, *the moment of inertia of a body with respect to any axis is equal to the moment of inertia with respect to a parallel axis through the centre of gravity, plus the mass of the body multiplied by the square of the distance between the two axes.*

Put $C =$ the moment of inertia with respect to an axis through the centre of gravity; then the distance from the axis of suspension to the centre of oscillation, the axes being parallel, will be

$$l = \frac{C + a^2\,M}{M\,k} \tag{3}$$

11. Examples.—

1. Find the centre of oscillation of a slender rod or straight line suspended at any point.

Let a and b be the lengths on opposite sides of the axis of suspension, then by (1)

$$l = \frac{\int r^2\,d\,M}{M\,k} = \frac{\int r^2\,d\,r}{(a + b)\frac{1}{2}\,(a - b)} = \frac{2\,(a^3 + b^3)}{3\,(a^2 - b^2)} = \frac{2\,(a^2 - a\,b + b^2)}{3\,(a - b)}$$

between the limits $r = +\,a$ and $r = -\,b$.

If the rod is suspended at its extremity, $b = 0$, and $l = \frac{2}{3}\,a$. If it is suspended at its middle point, $a = b$ and $l = \infty$.

2. Find the centre of oscillation of an isosceles triangle vibrating about an axis in its own plane passing through its vertex.

Put b and h for the base and altitude of the triangle; then by

(1), $l = \dfrac{\int_0^h r^2 \cdot \frac{b}{h}\,r\,d\,r}{\frac{1}{2}\,b\,h \cdot \frac{2}{3}\,h} = \frac{3}{4}\,h.$

If the axis of suspension coincides with the base of the triangle, then $l = \dfrac{\int_0^h r^2 \cdot \frac{b}{h}\,(h - r)\,d\,r}{\frac{1}{2}\,b\,h \cdot \frac{1}{3}\,h} = \dfrac{h}{2}.$

3. Find the centre of oscillation of a circle vibrating about an axis in its own plane.

$$C = \int r^2\,d\,M = 2 \int x^2\,y\,d\,x = 2 \int x^2\,(R^2 - x^2)^{\frac{1}{2}}\,d\,x =$$
$$-\,x\,\frac{(R^2 - x^2)^{\frac{3}{2}}}{2} + \frac{R}{2} \int (R^2 - x^2)^{\frac{1}{2}}\,d\,x.$$

Taking this integral between $x = -r$ and $x = +r$, we have

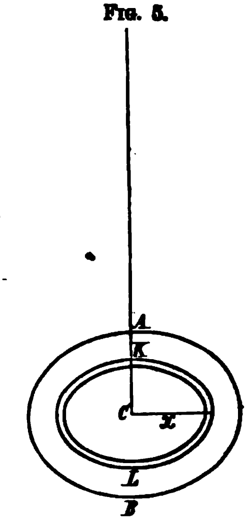
$$C = \frac{R^2}{2} \cdot \frac{\pi R^2}{2} = \frac{\pi R^4}{4}.$$

Substituting this value of C in (3) we have

$$l = \frac{\frac{\pi R^4}{4} + a^2 \pi R^2}{a \pi R^2} = a + \frac{R^2}{4a}.$$

4. Find the centre of oscillation of a circle vibrating about an axis perpendicular to it.

Fig. 5.



Let $K L$ (Fig. 5) be an elementary ring whose radius is x and whose breadth is $d x$; then

$$d M = 2 \pi x \, d x, \text{ and } C = \int_0^R x^2 \, . \, 2 \pi x \, d x$$

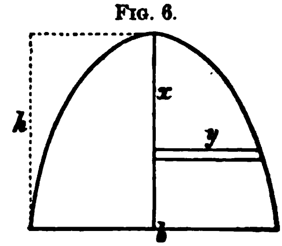
$$= \frac{\pi R^4}{2}; \therefore l = \frac{\frac{\pi R^4}{2} + a^2 \pi R^2}{\pi R^2 a} = a + \frac{R^2}{2a}.$$

As $a + \frac{R^2}{2a}$ is greater than $a + \frac{R^2}{4a}$, a circular pendulum will vibrate faster when the axis of suspension is in its plane, than when it is perpendicular to it.

IV. Centre of Hydrostatic Pressure.

12. General Formula.—Let the surface pressed upon be plane and vertical; and let the water level be the plane of reference. Suppose the surface to have a symmetrical form with reference to a vertical axis, x, whose ordinate is y (Fig. 6). A horizontal element of the surface is $2 \, y \, d \, x$, and (since the pressure varies as the depth) the pressure on that element $2 \, x \, y \, d \, x$. Hence the whole pressure to the depth x is $\int 2 \, x \, y \, d \, x = 2 \int x \, y \, d \, x$. The moment of the pressure on the element of surface is $2 \, x^2 \, y \, d \, x$; and the sum of all the moments to the same depth is $\int 2 \, x^2 \, y \, d \, x = 2 \int x^2 \, y \, d \, x$. Therefore, putting p for the depth of the centre of pressure, $p = \frac{\int x^2 \, y \, d \, x}{\int x \, y \, d \, x}$.

Fig. 6.



13. Examples.

1. *A rectangle.*—Let its height $= h$, and its base $= b$; then $2y$ everywhere equals b, and a horizontal element at the depth x is $b\,d\,x$, the pressure on it is $b\,x\,d\,x$, and the moment of that pressure is $b\,x^2\,d\,x$; $\therefore$ the depth of the centre of pressure $p = \frac{\int b\,x^2\,d\,x}{\int b\,x\,d\,x} = \frac{\frac{1}{3}b\,x^3 + c}{\frac{1}{2}b\,x^2 + c'}$. Since the pressure and area is each zero, when x is zero, c and c' both disappear, and $p = \frac{2}{3}x$, which for the whole surface becomes $p = \frac{2}{3}h$. That is, the centre of pressure on a vertical rectangular surface reaching to the water level, is two-thirds of the distance from the middle of the upper side to the middle of the lower.

2. *A triangle whose vertex is at the surface of the water, and its base horizontal.*—Let the triangle be isosceles, its height $= h$, and its base $= b$; then $h : b :: x : 2\,y = \frac{b}{h}\,x$. Therefore $p = \frac{\int \frac{b}{h}\,x^3\,d\,x}{\int \frac{b}{h}\,x^2\,d\,x}$ $= \frac{\frac{1}{4}x^4}{\frac{1}{3}x^3} = \frac{3}{4}\,x$; and for the whole height, $\frac{3}{4}\,h$.

If the triangle is not isosceles, it may be easily shown that the centre of pressure is on the line joining the vertex and the middle of the base, at a distance from the vertex equal to three-fourths of the length of that line.

3. *A triangle whose base is at the water level.*—Then $h : b : h - x : 2\,y = b - \frac{b}{h}\,x$. Therefore the pressure is $\int - b\,x\,d\,x - \int - \frac{b}{h}\,x^2\,d\,x$, because $d\,x$ is negative. The moment of the pressure is $\int - b\,x^2\,d\,x - \int - \frac{b}{h}\,x^3\,d\,x$.

$$\text{Therefore } p = \frac{-\int b\,x^2\,d\,x + \int \frac{b}{h}\,x^3\,d\,x}{-\int b\,x\,d\,x + \int \frac{b}{h}\,x^2\,d\,x} = \frac{-\frac{1}{3}\,x^3 + \frac{1}{4\,h}x^4}{-\frac{1}{2}\,x^2 + \frac{1}{3\,h}x^3} =$$

$\frac{4\,h\,x^3 - 3\,x^4}{6\,h\,x^2 - 4\,x^3} = \frac{4\,h\,x - 3\,x^2}{6\,h - 4x}$; and, when $x = h$, this becomes $\frac{1}{2}\,h$. In general, the centre of pressure is at the middle of the line joining the vertex and the middle of the base.

4. *A parabola whose vertex is at the surface.*—As $y = p^{\frac{1}{2}}\,x^{\frac{1}{2}}$, therefore $p = \frac{\int x^2\,p^{\frac{1}{2}}\,x^{\frac{1}{2}}\,d\,x}{\int x\,p^{\frac{1}{2}}\,x^{\frac{1}{2}}\,d\,x} = \frac{\int x^{\frac{5}{2}}\,d\,x}{\int x^{\frac{3}{2}}\,d\,x} = \frac{\frac{2}{7}\,x^{\frac{7}{2}}}{\frac{2}{5}\,x^{\frac{5}{2}}} = \frac{5}{7}x$; or $\frac{5}{7}\,h$, for the whole area.

5. *A parabola whose base is at the surface.*—As $h - x$ is the depth of an element, $d\,x$ is negative. $p = \dfrac{-\int (h-x)^2 x^{\frac{1}{2}} d\,x}{-\int (h-x)\, x^{\frac{1}{2}} d\,x} =$

$$\frac{\int (h^2 x^{\frac{1}{2}}\, d\,x - 2\,h\,x^{\frac{3}{2}}\, d\,x + x^{\frac{5}{2}}\, d\,x)}{\int (h\ x^{\frac{1}{2}}\, d\,x - x^{\frac{3}{2}}\, d\,x)} = \frac{\frac{2}{3} h^2 x^{\frac{3}{2}} - \frac{4}{5} h\, x^{\frac{5}{2}} + \frac{2}{7} x^{\frac{7}{2}}}{\frac{2}{3} h\, x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}}} =$$

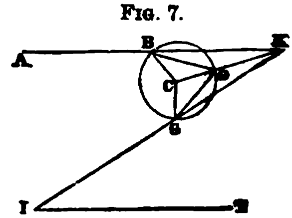
$\dfrac{\frac{1}{3} h^2 - \frac{2}{5} h\,x + \frac{1}{7} x^2}{\frac{1}{3} h - \frac{1}{5} x}$; and when $x = h$, the expression becomes

$$\frac{\frac{1}{3} h^2 - \frac{2}{5} h^2 + \frac{1}{7} h^2}{\frac{1}{3} h - \frac{1}{5} h} = \frac{4}{7} h.$$

V. Angular Radius of the Primary and Secondary Rainbow and the Halo.

14. The Primary Rainbow.—Since the primary bow is formed by those rays which, on emerging after one reflection, make the largest angle with the incident rays, proceed to find what angle of incidence will cause the largest deviation of the emerging rays.

Fig. 7.



In Fig. 7, let x = angle of incidence; y = angle of refraction; z = angle of deviation; n = index of refraction. Then, in the quadrilateral $B\,D\,G\,K$, $D\,B\,K = D\,G\,K = x - y$; angle at $D = 360 - 2\,y$; $\therefore K = z = 4\,y - 2\,x$;

$$\therefore \frac{d\,z}{d\,x} = \frac{4\,d\,y}{d\,x} - 2 = 0.$$

But $\qquad \sin x = n \sin y;$

$$\therefore \cos x\, d\,x = n \cos y\, d\,y, \text{ and } \frac{d\,y}{d\,x} = \frac{\cos x}{n \cos y}.$$

By substitution, $\dfrac{4 \cos x}{n \cos y} = 2.$

$$\therefore 2 \cos x = n \cos y; \text{ and } 4 \cos^2 x = n^2 \cos^2 y.$$

But $\qquad \sin^2 x = n^2 \sin^2 y;$

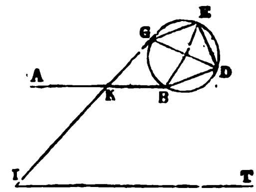
$$\therefore 3 \cos^2 x + 1 = n^2; \text{ since } \sin^2 + \cos^2 = 1.$$

$$\therefore \cos x = \sqrt{\frac{n^2 - 1}{3}}$$

If 1.33 and 1.55, the values of n for extreme red and violet, be used in this formula, we obtain x, and therefore y and z, for the limiting angles of the primary bow.

15. The Secondary Bow.—To find the angle of minimum deviation. Using the same notation as before, we have in the pentagon $GEDBK$ (Fig. 8), $G = B = 180 - x + y$; $E = D = 2y$; $\therefore K = z = 180 + 2x - 6y$;

FIG. 8.



$$\therefore \frac{dz}{dx} = 2 - \frac{6\,dy}{dx} = 0.$$

$$\therefore \frac{6 \cos x}{n \cos y} = 2; \text{ and } 3 \cos x = n \cos y;$$

$$\therefore 9 \cos^2 x = n^2 \cos^2 y;$$

but $\sin^2 x = n^2 \sin^2 y$;

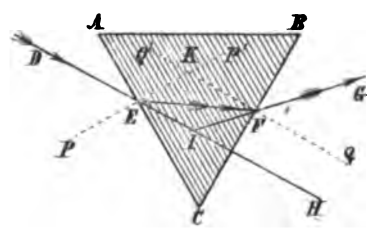
$$\therefore 8 \cos^2 x + 1 = n^2;$$

$$\therefore \cos x = \sqrt{\frac{n^2 - 1}{8}};$$

which, as before, will furnish z for each limiting color of the secondary bow.

16. The Common Halo.—Let DE (Fig. 9) be the ray from the sun, and FG the emergent ray. Let $DEp = x$; $KEF = y$; $KFE = x'$; $GFp = y'$; $I = z = x - y + y' - x'$. Now, $y + x' = p'KF = C = 60°$.

FIG. 9.



$$\therefore z = x + y' - C.$$

$\sin x = n \sin y$,

and $\sin y' = n \sin x'$;

$$\therefore x = \sin^{-1} (n \sin y),$$

and $y' = \sin^{-1} (n \sin x') = \sin^{-1} \{n \sin (C - y)\}$.

By substitution,

$z = \sin^{-1} (n \sin y) + \sin^{-1} \{n \sin (C - y)\} - C$. Therefore z is a function of y; and, by differentiating, we have

$$\frac{dz}{dy} = \frac{n \cos y}{\sqrt{1 - n^2 \sin^2 y}} - \frac{n \cos (C - y)}{\sqrt{1 - n^2 \sin^2 (C - y)}} = 0.$$

$$\therefore \frac{n^2 \cos^2 y}{1 - n^2 \sin^2 y} = \frac{n^2 \cos^2 (C - y)}{1 - n^2 \sin^2 (C - y)};$$

$$\therefore \frac{1 - \sin^2 y}{1 - n^2 \sin^2 y} = \frac{1 - \sin^2 (C - y)}{1 - n^2 \sin^2 (C - y)};$$

$$\therefore (n^2 - 1) \sin^2 y = (n^2 - 1) \sin^2 (C - y);$$

$$\therefore y = C - y, \text{ and } y = \tfrac{1}{2} C;$$

$$\text{and } x' = \tfrac{1}{2} C.$$

Hence, the minimum deviation occurs when the ray within the crystal is equally inclined to the sides. Knowing n, the index of refraction for ice, x, and its equal, y', can be obtained, and then z, the deviation required.